

DIRECTIONS  This exam has two parts. Part A has 6 short answer questions (7 points each, so 42 points) while Part B has 4 traditional problems (15 points each, so 60 points). Total: 102 points. Neatness counts.

Closed book, no calculators, computers, ipods, cell phones, etc – but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: six short answer questions (7 points each, so 42 points).

1. Find a $3 \times 3$ symmetric matrix $A$ with the property that
   \[
   \langle X, AX \rangle = -x_1^2 + 6x_1x_2 - x_1x_3 + 2x_2x_3 + 3x_2^2
   \]
   for all $X = (x_1, x_2, x_3) \in \mathbb{R}^3$.

2. Under what conditions on the constants $a$, $b$, $c$, and $d$ is the following matrix $A$ positive definite?
   \[
   A := \begin{pmatrix}
   a & 0 & 0 & 0 \\
   0 & b & 0 & 0 \\
   0 & 0 & c & 0 \\
   0 & 0 & 0 & d
   \end{pmatrix}
   \]

3. Let $B$ be an anti-symmetric $n \times n$ real matrix, so $B^* = -B$. Show that $\langle V, BV \rangle = 0$ for all $V \in \mathbb{R}^n$.

4. Find the arc length of the segment of the helix $X(t) := (\cos 3t, 1 - 4t, \sin 3t)$, for $0 \leq t \leq \pi$.

5. Find some function $u(x, y)$ that satisfies $\frac{\partial^2 u}{\partial x \partial y} = 4 \cos(x + 2y) - 2xy$.

6. Let $v(s)$ be a smooth function of the real variable $s$ and let $u(x, t) := v(x + 3t)$. Show that $u$ satisfies the homogeneous partial differential equation $u_t - 3u_x = 0$.

[Part B is on the next page]
PART B: four traditional problems (15 points each, so 60 points).

B–1. In an experiment, at time $t$ you measure the value of a quantity $R$ and obtain:

<table>
<thead>
<tr>
<th>$t$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-3</td>
</tr>
</tbody>
</table>

Based on other information, you believe the data should fit a curve of the form $R = a + bt^2$.

a) Write the (over-determined) system of linear equations you would ideally like to solve for the unknown coefficients $a$ and $b$.

b) Use the method of least squares to find the normal equations for the coefficients $a$ and $b$.

c) Solve the normal equations to find the coefficients $a$ and $b$.

B–2. Find and classify all the critical points of $f(x, y, z) := x^3 - 3x + y^2 + z^2$.

B–3. For a certain rod of length $\pi$, the temperature $u(x, t)$ at the point $x$ at time $t$ satisfies the heat equation $u_t = u_{xx}$. Find all solutions of the special form

$$u(x, t) = w(x)T(t) \quad \text{for} \quad 0 \leq x \leq \pi$$

that satisfy the boundary conditions $u(0, t) = 0$ and $u(\pi, t) = 0$ for all $t \geq 0$.

B–4. Say the equation $f(X) := f(x, y, z) = 0$ implicitly defines a smooth surface in $\mathbb{R}^3$ (an example is the sphere $x^2 + y^2 + z^2 - 4 = 0$). Let $P \in \mathbb{R}^3$ be a point not on this surface. Assume $Q$ is a point on the surface that is closest to $P$. Show that the vector from $P$ to $Q$ is orthogonal to the tangent plane to the surface at $Q$.

[SUGGESTION: Let $X(t)$ be a smooth curve in the surface with $X(0) = Q$. Then $Q$ is the point on the curve that is closest to $P$.]