Math 260  
Final Exam  
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May 2, 2012  
9:00 – 11:00

**Directions**  
This exam has two parts. **Part A** has 5 short answer questions (5 points each, so 25 points) while **Part B** has 8 traditional problems (10 points each, so 80 points). Total: 105 points. *Neatness counts.*

Closed book, no calculators, computers, ipods, cell phones, etc – but you may use one 3" x 5" card with notes on both sides.

**Part A:** Five short answer questions (5 points each, so 25 points).

**A-1.** Let $S$ be the linear space of $2 \times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$. Compute the dimension of $S$.

**A-2.** Let $V$ and $W$ be linear spaces and $L : V \to W$ a linear map. Let $w_1$ and $w_2$ be in $W$. Say $v_1 \in V$ is a solution of $L v_1 = w_1$ while both $v_2$ and $v_3$ are distinct points in $V$ that satisfy $L v_2 = L v_3 = w_2$. Does the equation $L x = w_1$ have a solution other than $v_1$? Explain your reasoning.

**A-3.** Let $f(t)$ be a smooth function of the real variable $t$. Show that for any real constants $a$ and $b$, the function $u(x, y) := f(ax + by)$ satisfies $u_{xx} u_{yy} - u_{xy}^2 = 0$.

**A-4.** Consider the surface defined implicitly by $x^2 + 9y^2 - z^2 = 10$. Find a vector orthogonal to the tangent plane at $(1, 1, 0)$.

**A-5.** Let $J := \int_0^2 \left( \int_0^x f(x, y) dy \right) dx$. Rewrite this as an iterated integral with the order of integration reversed, so one first integrates with respect to $x$.

**Part B:** Eight traditional problems (10 points each, so 80 points).

**B-1.** Consider the set of real-valued continuous functions on the interval $-1 \leq x \leq 1$ with the inner product $\langle f, g \rangle := \int_{-1}^1 f(x)g(x) dx$.

a) Find a quadratic polynomial $p(x) := a + bx + cx^2$ (with $a \neq 0$) that is orthogonal to both $e_1(x) := 1$ and $e_2(x) := x$.

b) Find the orthogonal projection of $q(x) := x^4$ into the subspace spanned by $e_1(x)$, $e_2(x)$, and $p(x)$.

**B-2.** Find a solution of $u'' + 4u = x^2$ that satisfies the initial conditions $u(0) = 0$ and $u'(0) = 0$. 

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B-3. Let \( A \) be a real \( n \times n \) antisymmetric matrix.

a) Show that \( \langle X, AX \rangle = 0 \) for all vectors \( X \in \mathbb{R}^n \).

b) Say \( X(t) \) is a solution of the differential equation \( \frac{dX}{dt} = AX \). Show that \( \|X(t)\| = \) constant. [Remark: In the special case \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) this implies \( \sin^2 t + \cos^2 t = 1 \).]

B-4. Find and classify the critical points of \( g(x,y) := x^2 - 2xy + \frac{1}{3}y^3 - 3y \).

B-5. Compute \( \oint_{\gamma} 2x \, dy - y \, dx \) where the closed curve \( \gamma \) is the triangle in \( \mathbb{R}^2 \) with vertices at \( (0,0) \), \( (1,0) \), and \( (1,2) \), traversed counterclockwise.

B-6. Let \( V = (y^2 + x)\mathbf{i} + (2xy - 3)\mathbf{j} \).

a) Find a function \( u(x,y) \) so that \( V = \nabla u \).

b) Let \( \gamma \) be the triangle bounded by the \( x \)-axis, the \( y \)-axis, and the straight line \( 2x + y = 2 \), traversed counterclockwise. Compute \( \oint_{\gamma} V \cdot ds \).

B-7. Consider the region \( \Omega \subset \mathbb{R}^3 \) above the surface \( z = x^2 + y^2 \) and below the plane \( z = 4 \).

Compute \( \iiint_{\Omega} 2z \, dV \).

B-8. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with smooth boundary \( \partial \Omega \) and let \( w(x,y,t) \) be the solution of the heat equation

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 w_t = \Delta w \quad \text{for all } (x,y) \in \Omega \quad \text{and } t \geq 0, \quad \text{with } w = 0 \quad \text{for } (x,y) \text{ on } \partial \Omega.
\]

a) Define \( E(t) := \frac{1}{2} \iiint_{\Omega} w^2(x,y,t) \, dx \, dy \). Show that \( dE/dt \leq 0 \).

b) If in addition the initial temperature \( w(x,y,0) = 0 \), show that \( w(x,y,t) = 0 \) for all \( (x,y) \in \Omega \) and \( t \geq 0 \).

c) If \( u(x,y,t) \) and \( v(x,y,t) \) both satisfy the heat equation in \( \Omega \) with \( u(x,y,t) = v(x,y,t) \) on \( \partial \Omega \) for all \( t \geq 0 \) and also \( u(x,y,0) = v(x,y,0) \), show that \( u(x,y,t) = v(x,y,t) \) for all \( (x,y) \in \Omega \) and \( t \geq 0 \).