Problem Set 11
Due: Thurs. April 5. Late papers will be accepted until 1:00 PM Friday.

1. Let $u(x, t) := \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds$. Show that $u$ satisfies the wave equation (for a vibrating string) $u_{tt} = c^2 u_{xx}$ with initial position $u(x, 0) = f(x)$ and initial velocity $u_t(x, 0) = g(x)$.

[Suggestion: If $G(p, q) := \int_p^q h(s) \, ds$, compute $\frac{\partial G(p, q)}{\partial p}$ and $\frac{\partial G(p, q)}{\partial q}$.

2. Let $Q \subset \mathbb{R}^3$ be the portion of the shell $1 \leq x^2 + y^2 + z^2 \leq 9$ that is in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$.
   a) Set up and evaluate the triple integral to compute the volume of $Q$.
   b) Compute $\iiint_Q z \, dV$.

3. [Marsden-Tromba, p.337 #2] Assuming uniform density, find the coordinates of the center of mass of the semicircle $y = \sqrt{r^2 - x^2} \geq 0$.

4. [Marsden-Tromba, p.337 #4] Find the average of $e^{x+y}$ over the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$.

5. [Marsden-Tromba, p.338 #16] Find the average of $e^{-z}$ over the unit ball $x^2 + y^2 + z^2 \leq 1$.

   a) Let $D$ be a region in the part of the $xy$-plane with $x > 0$. Assume $D$ has uniform density. Let $A(D)$ be its area and $(\bar{x}, \bar{y})$ its center of mass. Let $W$ be the solid obtained by rotating $D$ about the $y$-axis. Show that $\text{Vol}(W) = 2\pi \bar{x} A(D)$. [Note that $2\pi \bar{x}$ is the length of the circle traversed by the center of mass.]
   b) Apply this to compute the volume of the torus obtained by rotating the unit circle centered at $(3, 0)$ around the $y$-axis.

7. [Marsden-Tromba, P. 373 #3] Let $\mathbf{F}(x, y, z) = xi + yj + zk$. Evaluate the line integral of $\mathbf{F}$ along each of the following paths:
   a). $c(t) = (t, t, t), \quad 0 \leq t \leq 1$  
   b). $c(t) = (\cos t, \sin t, 0), \quad 0 \leq t \leq 2\pi$  
   c). $c(t) = (\sin t, 0, \cos t), \quad 0 \leq t \leq 2\pi$  
   d). $c(t) = (t^2, 3t, 2t^3), \quad -1 \leq t \leq 2$

8. Repeat the previous problem with $\mathbf{F}(x, y, z) = yi - xj + zk$.  

1
9. [Marsden-Tromba, P.374 #13] Let $c(t)$ be a path and $T$ the unit tangent vector. What is $\int_c T \cdot ds$?

10. Let $r := x\mathbf{i} + y\mathbf{j}$ and $V(x, y) := p(x, y)\mathbf{i} + q(x, y)\mathbf{j}$ be (smooth) vector fields and $C$ a smooth curve in the plane. In this problem $J$ is the line integral $J = \int_C \mathbf{V} \cdot d\mathbf{r}$. For each of the following, either give a proof or give a counterexample.
   a) If $C$ is a vertical line segment and $q(x, y) = 0$, then $J = 0$.
   b) If $C$ is a circle and $q(x, y) = 0$, then $J = 0$.
   c) If $C$ is a circle centered at the origin and $p(x, y) = -q(x, y)$, then $J = 0$.
   d) If $p(x, y) > 0$ and $q(x, y) > 0$, then $J > 0$.

11. Let $\Omega \subset \mathbb{R}^3$ be the half-ball where $x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$. Use symmetry to deduce (without computation) that $\iiint_\Omega x^3 dV = 0$.

12. If $\mathbf{F} := F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a conservative field, then $\mathbf{F} = \nabla u(x, y, z)$ for some scalar-valued function $u$ and $u$ is called the potential function for the field $\mathbf{F}$.
   a) If $\mathbf{F}$ is conservative, show that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, and $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$.
   b) Show that $\mathbf{F} := 2xi + zj + 2yk$ is not conservative.
   c) Show that $\mathbf{F} := 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$ is conservative by finding the potential function $u(x, z)$.
   d) Show that $\mathbf{F} := 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ is conservative by finding the potential function $u$.

13. [Marsden-Tromba, P.374 #17] Evaluate $\int_C 2xyz \, dx + x^2z \, dy + x^2z \, dz$ where $C$ is an oriented simple curve connecting $(1, 1, 1)$ to $(1, 2, 4)$.

14. [Marsden-Tromba, P.374 #18] Suppose $\nabla f(x, y, z) = 2xyze^{x^2}\mathbf{i} + ze^{x^2}\mathbf{j} + ye^{x^2}\mathbf{k}$. If $f(0, 0, 0) = 5$, find $f(1, 1, 2)$. 

2
**Bonus Problems**

[Please give this directly to Professor Kazdan]

B-1 Let $V_n(R)$ be the “volume” of the ball $B_n(R) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 \leq R^2 \}$ and $A_{n-1}(R)$ the “area” of the sphere $S^{n-1}$ of radius $R$ in $\mathbb{R}^n$, so $x_1^2 + \cdots + x_n^2 = R^2$.

Complete the outline begun in class to obtain the recursion formula

$$A_n(1) = \frac{2\pi}{n-1} A_{n-2}(1)$$

and use this to find formulas for $V_n(R)$ and $A_{n-1}(R)$. There are two cases depending if $n$ is even or odd.

Show that $\lim_{n \to \infty} V_n(1) = 0$.

B-2 Let $\Omega \subset \mathbb{R}^2$ be a connected open set. If $u(x, y)$ is a smooth scalar-valued function with the property that $\nabla u = 0$ throughout $\Omega$, show that $u(x, y)$ must be a constant.

[Last revised: April 1, 2012]