Problem Set 12

Due: never

Unless otherwise stated use the standard Euclidean norm.

1. Let \( \gamma(t) \) be any smooth closed curve in \( \mathbb{R}^4 \). Why, with only a mental computation, is

\[
\oint_{\gamma} 2x \, dx + 6(x - y) \, dy = \oint_{\gamma} 6x \, dy
\]

2. Find some closed curve \( \gamma(t) \) so that \( \oint_{\gamma} 6x \, dy > 0 \).

3. Let \( C \) be the portion of the unit circle \( x^2 + y^2 = 1 \) with \( x \geq 0 \) oriented so that it begins at \((0, 1)\) and ends at \((0, -1)\). Evaluate

\[
\int_{\gamma} e^x \sin y \, dx + e^x \cos y \, dy.
\]

4. Let \( \mathbf{F} \) be a continuous force field defined on \( \mathbb{R}^3 \) and suppose that a particle of mass \( m \) moves along a path \( X(t) \) determined by Newton’s second law of motion, \( mX'' = \mathbf{F}(X(t)) \) during the time interval \( a \leq t \leq b \). Show that

\[
\int_{a}^{b} \mathbf{F} \cdot X'(t) \, dt = \frac{m}{2} \|X'(b)\|^2 - \frac{m}{2} \|X'(a)\|^2.
\]

In physics, the right hand side is interpreted as a change in kinetic energy.

5. Let \( \psi(t) \) be a scalar-valued function with a continuous derivative for \( 0 < t < \infty \) and let \( \mathbf{X} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \in \mathbb{R}^3 \). Define the vector field \( \mathbf{F}(x, y, z) := \psi(\|\mathbf{X}\|)\mathbf{X} \) for all \( \mathbf{X} \neq 0 \). Show that this vector field is conservative by finding a scalar-valued function \( \varphi(r) \) with the property that \( \mathbf{F}(\mathbf{X}) := \nabla \varphi(\|\mathbf{X}\|) \). In particular, this shows that every central force field is conservative except possibly at the origin.

6. [Marsden-Tromba p. 381 #2] A surface in \( \mathbb{R}^3 \) is defined by \( x = u^2 - v^2, \ y = u + v, \ z = u^2 + 4v \).

   a) At what points is this surface regular?
   
   b) Find the equation of the tangent plane at \((\frac{3}{4}, \frac{1}{2}, 2)\) (so \( u = 0, \ v = \frac{1}{2} \)).

7. [Marsden-Tromba p. 381 #7] Match the parametrization as belonging to the surfaces: (i) ellipsoid, (ii) parabolic cylinder, (iii) hyperboloid, or (iv) cone. [Corresponding drawings are in the text].
a) \[ \Phi(u, v) := ((2\sqrt{1+u^2}) \cos v, (2\sqrt{1+u^2}) \sin v, u) \]
b) \[ \Phi(u, v) := (3 \cos u \sin v, 2 \sin u \sin v, \cos v) \]
c) \[ \Phi(u, v) := (u, v, u^2) \]
d) \[ \Phi(u, v) := (u \cos v, u \sin v, u) \]

8. [Marsden-Tromba p.383 #18] For a sphere in \( \mathbb{R}^3 \) centered at the origin with radius 2, find the equation of the tangent plane at the point \((1, 1, 1/\sqrt{2})\) by considering the sphere as
a) a surface parametrized by \( \Phi(\theta, \phi) := (2 \cos \theta \sin \phi, 2 \sin \theta \sin \phi, 2 \cos \phi) \),
b) a level surface of \( f(x, y, z) := x^2 + y^2 + z^2 \),
c) the graph of \( g(x, y) := \sqrt{4 - x^2 - y^2} \).

9. We used the following parametrization of the torus:
\[ x(\theta, \phi) = (3 + \cos \phi) \cos \theta, \quad y(\theta, \phi) = (3 + \cos \phi) \sin \theta, \quad z(\theta, \phi) = \sin \phi, \]
where \( 0 \leq \theta, \phi \leq 2\pi \). Show that the image surface (our torus) is regular at all points.

10. Compute the area of the torus using the parametrization of the above problem.

11. [Marsden-Tromba p.392 #26d] Consider the graph of \( z := y^3 \cos^2 x \) over the triangle with vertices at \((-1, 1), (0, 2), (1, 1)\). Express the surface area as a double integral (but don’t evaluate it).

12. [Marsden-Tromba p.398 #4] Evaluate the integral \( \iint_S (x + z) \, dS \), where \( S \) is the part of the cylinder \( y^2 + z^2 = 4 \) with \( 0 \leq x \leq 5 \).

13. [Similar to Marsden-Tromba p.399 #23] Let \( S \) be a two dimensional surface in \( \mathbb{R}^n \), \( n \geq 3 \), given by the parametrization \( (u, v) \mapsto \Phi(u, v) \) with
\[ x_1 = x_1(u, v), \quad x_2 = x_2(u, v), \quad \ldots \quad x_n = x_n(u, v). \]
Say we have a curve \( \gamma(t) = (u(t), v(t)) \). Then its image under \( \Phi \) gives a curve \( X(t) := \Phi(u(t), v(t)) \) in the surface in \( \mathbb{R}^n \). As usual, the element of arc length is given by \( ds = \|\gamma'(t)\|dt \). Show that
\[
\left( \frac{ds}{dt} \right)^2 = E(u, v) \left( \frac{dv}{dt} \right)^2 + 2F(u, v) \frac{dv}{dt} \frac{du}{dt} + G(u, v) \left( \frac{du}{dt} \right)^2, \tag{1}
\]
where
\[ E(u, v) = \left\| \frac{\partial \Phi}{\partial u} \right\|^2, \quad F(u, v) = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v}, \quad G(u, v) = \left\| \frac{\partial \Phi}{\partial v} \right\|^2. \]

We think of the formula (11) as defining an inner product on tangent vectors and use the symmetric matrix
\[ g := \begin{pmatrix} E(u, v) & F(u, v) \\ F(u, v) & G(u, v) \end{pmatrix}. \]

Since the element of arc length is always positive on non-trivial curves, this matrix \( g \) is required to be positive definite. The element of area on the surface is \( dS := \sqrt{\det g} \, du \, dv = \sqrt{EG - F^2} \, du \, dv. \) Note that this works in any dimension (the cross product version works only in dimension \( n = 3 \)).

We often write equation (11) as
\[ ds^2 = E(u, v) \, du^2 + 2F(u, v) \, du \, dv + G(u, v) \, dv^2 \]
and refer to it as specifying a Riemannian Metric on the surface.

[Last revised: April 11, 2012]