Problem Set 13  
Due: Thursday April 26, 1 PM

Unless otherwise stated use the standard Euclidean norm. Also all regions \( \Omega \subset \mathbb{R}^n \) are assumed to be bounded, connected, and have smooth boundaries.

Remark: The first two problems were originally on Exam 3, but at the last moment I deleted them fearing the exam was too long.

1. Let \( \mathbf{V} = (y^2 + 1)i + (2xy - 4y)j + 2k \)
   a) Find a function \( u(x, y, z) \) so that \( \mathbf{V} = \nabla u \).
   b) Let \( \gamma \) be the triangle bounded by the \( x \)-axis, the \( y \)-axis, and the straight line \( 2x + y = 2 \), traversed counterclockwise. Compute \( \oint_\gamma \mathbf{V} \cdot d\mathbf{s} \).

2. a) Let \( \Omega \subset \mathbb{R}^3 \) be the region below the surface \( z = 4 - (x^2 + y^2) \) and above the \( xy \)-plane. Compute \( \iiint_{\Omega} z \, dV \).
   b) Let \( \Omega \subset \mathbb{R}^3 \) be the region below the surface \( z = 4 - (x^2 + 4y^2) \) and above the \( xy \)-plane. Compute \( \iiint_{\Omega} z \, dV \).

3. Compute \( \oint_{\gamma} x \, dy - y \, dx \) where the closed curve \( \gamma \) is the triangle in \( \mathbb{R}^2 \) with vertices at \((0, 0)\), \((1, 0)\), and \((1, 2)\), traversed counterclockwise.

4. [Marsden-Tromba, p. 437 # 6] Verify the Green’s-Stokes’ theorem in the plane \( \oint_D P \, dx + Q \, dy = \iint_D \text{curl} \, \mathbf{F} \) for the region \([0, \frac{\pi}{2}]\), \([0, \frac{\pi}{2}]\), with \( P(x, y) = \sin x \), \( Q(x, y) = \cos y \). You should compute both sides of the formula to verify that they agree.

5. [Marsden-Tromba p. 437 # 11d]. Verify the Green’s-Stokes’ theorem in the plane or the disk \( D \) with center at the origin and radius \( R \) for \( P(x, y) = 2y \), \( Q(x, y) = x \).

6. [Marsden-Tromba p. 437 # 15]. Evaluate \( \int_{C} (2x^3 - y^3) \, dx + (x^3 + y^3) \, dy \) where \( C \) is the unit circle both directly and using the Green’s-Stokes’ theorem in the plane.

7. [Marsden-Tromba p. 438 # 20]. Let \( P(x, y) = \frac{-y}{(x^2 + y^2)} \) and \( Q(x, y) = x/(x^2 + y^2) \) in the unit disc \( D \). Show that Green’s theorem fails for this \( P \) and \( Q \). Explain why.
8. [Marsden-Tromba p. 439 # 38]. Use Green’s theorem in the plane to prove the change of variables formulas in the following special case

\[
\iint_D dx
dy = \iint_{D'} \left| \frac{\partial(x,y)}{\partial(u,v)} \right|
du
dv
\]

for a transformation \((u, v) \mapsto (x(u,v), y(u,v))\).

9. In applying the divergence theorem where the region is all of \(\mathbb{R}^3\), the integral over the boundary is not well defined. Instead, one works on the ball of radius \(R\) and then lets \(R \to \infty\).

Suppose \(V(x,y,z)\) is a vector-valued function defined everywhere in 3-dimensional space. Further, suppose that \(V\) is differentiable and that for some constant \(c\)

\[
\|V(x,y,z)\| \leq \frac{c}{1 + (x^2 + y^2 + z^2)^{3/2}}
\]

for all \((x,y,z)\). Show that

\[
\iiint_{\mathbb{R}^3} \nabla \cdot V(x,y,z)\,dx\,dy\,dz = 0. \tag{1}
\]

In other words, if \(B(0,R)\) is the ball of radius \(R\) centered at the origin, then (1) means that

\[
\lim_{R \to \infty} \iiint_{B(0,R)} \nabla \cdot V(x,y,z)\,dx\,dy\,dz = 0.
\]

10. a) Say \(u(x)\) satisfies \(u'' - c(x)u = 0\) on the bounded interval \(a < x < b\) with \(u(x) = 0\) on the boundary, so \(u(a) = 0\) and \(u(b) = 0\). Assuming that \(c(x) \geq 0\), show that then the only possibility is \(u(x) = 0\) throughout the interval. [Suggestion: Multiply the equation by \(u\) and integrate over the interval. Then integrate by parts.] The example \(u'' + u = 0\) on \(0 < x < \pi\), one of whose solutions is \(\sin x\) shows that the assumption \(c(x) \geq 0\) plays a vital role.

b) Say \(u(x,y)\) satisfies \(\Delta u - c(x,y)u = 0\) in a bounded region \(\Omega\) in the plane with \(u(x,y) = 0\) on the boundary, \(\partial\Omega\). Assuming that \(c(x,y) \geq 0\), show that then the only possibility is \(u(x,y) = 0\) throughout \(\Omega\).

c) Let \(u(x,y)\) and \(v(x,y)\) satisfy \(\Delta u - c(x,y)u = f(x,y)\) in \(\Omega\) with \(u(x,y) = \phi(x,y)\) on \(\partial\Omega\), as well as \(\Delta v - c(x,y)v = f(x,y)\) in \(\Omega\) with \(v(x,y) = \phi(x,y)\) on \(\partial\Omega\), so they satisfy the same differential equation and the same boundary condition. As above, assume \(c(x,y) \geq 0\). Show that \(u = v\) throughout \(\Omega\).

11. a) [Vibrating String] Let \(u(x,t)\) be a solution of the wave equation \(u_{tt} = u_{xx}\) in one space variable, say \(0 \leq x \leq L\). Assume the ends of the string are fixed:
$u(0, t) = 0$ and $u(L, t) = 0$. Define the energy as

$$E(t) := \frac{1}{2} \int_0^L [u_t^2 + u_x^2] \, dx.$$  

Show that energy is conserved: $dE/dt = 0$. [HINT: At some step of the computation integrate by parts using that because of the boundary condition, the velocity is zero at the end points.]

b) Use this to show that if the initial position and initial velocity are zero, so $u(x, y, 0) = 0$, $u_t(x, y, 0) = 0$, then $(x, y, t) = 0$ for all $(x, y) \in \Omega$ and all $t \geq 0$.

c) [Vibrating Drumhead] Let $u(x, y, t)$ be a solution of the wave equation $u_{tt} = u_{xx} - u_{yy}$ for $(x, y)$ in a bounded set $\Omega$ in $\mathbb{R}^2$ (the drumhead). Assume the drumhead is fixed along its boundary: $u(x, y, t) = 0$ for $(x, y) \in \partial \Omega$. Define the energy as

$$E(t) := \frac{1}{2} \iint_{\Omega} [u_t^2 + |\nabla u|^2] \, dxdy.$$  

Show that energy is conserved: $dE/dt = 0$.

**Bonus Problem**

[Please give these directly to Professor Kazdan]

**Notation:** Let $u(x, y)$ be a smooth function on the plane (actually, we will only use that the second derivatives are continuous) and $D \subset \mathbb{R}^2$ be an open region. Given a point $p \in D$, let $B_r(p)$ be the closed disk of radius $r$ centered at $p$ and contained in $D$ for $0 < r \leq R$ (so just pick $R$ sufficiently small). Define $I(r)$ by

$$I(r) := \frac{1}{2\pi r} \int_{\partial B_r(p)} u \, ds.$$  

This is just the average of $u$ on this circle.

B-1 [Marsden-Tromba p. 438–9 # 29-34]

a) Show that $\lim_{r \to 0} I(r) = u(p)$.

b) Let $n$ denote the unit outer normal to $\partial B_r$ and define $\partial u/\partial n := \nabla u \cdot n$ (this is the directional derivative of $u$ in the direction of the outer normal). Show that

$$\int_{\partial B_r} \frac{\partial u}{\partial n} \, ds = \iint_{B_r} \Delta u \, dA.$$  

c) Use this to show that $I'(r) = \frac{1}{2\pi r} \iint_{B_r} \Delta u \, dA$. 

d) Suppose that $u$ is a harmonic function, that is, $\Delta u = 0$ in $D$. Use the above to deduce the mean value property of harmonic functions

$$u(p) = \frac{1}{2\pi r} \int_{\partial B_r} u \, ds.$$ 

This states the the value of $u$ at the center of a disk is the average of its values on the circumference.

e) From the previous part, deduce the “solid mean value property”

$$u(p) = \frac{1}{\pi R^2} \int_{B_R} u \, dA.$$ 

f) If $u$ is harmonic in $D$ and has a local maximum at some point $p$ in $D$, show that $u$ must be a constant in some small disk centered at $p$.

g) Assuming that $D$ is connected, show that if $u$ is harmonic in $D$ and has its absolute maximum at some point $p$ in $D$ (so $u(p) \geq u(q)$ for all points $q \in D$), then $u$ must be a constant $D$.

Similarly, if $u$ has its absolute minimum at some point $p$ in $D$, then $u$ must be a constant in $D$.

[Last revised: May 10, 2012]