Problem Set 7

DUE: In class Thursday, March 1. Late papers will be accepted until 1:00 PM Friday.

Unless otherwise stated use the standard Euclidean norm.

1. Say you have a matrix \( A(t) = ((a_{ij}(t))) \) whose elements \( a_{ij}(t) \) depend on a parameter \( t \in \mathbb{R} \). We say that \( A(t) \) is differentiable at \( t_0 \) if the limit
   \[
   \lim_{h \to 0} \frac{A(t_0 + h) - A(t_0)}{h}
   \]
   exists. We then call this limit \( A'(t_0) \). It is easy to show that \( A(t) \) is differentiable if and only if all of the elements \( a_{ij}(t) \) are differentiable.
   a) Say \( A(t) \) and \( B(t) \) are \( n \times n \) matrices that are differentiable and let \( C(t) := A(t)B(t) \) Show that \( C(t) \) is differentiable and give a formula for \( C'(t) \) in terms of \( A, A', B, \) and \( B' \)
   b) If \( A(t) \) is differentiable and invertible, show that \( A^{-1}(t) \) is differentiable and give a formula for \( dA^{-1}(t)/dt \) in terms of \( A, A^{-1}, \) and \( A' \). [SUGGESTION: Imitate the proof for a \( 1 \times 1 \) matrix.]

2. In each of the following find \( u(x,t) \).
   a) \( \frac{\partial u}{\partial t} = 0 \) with \( u(x,0) = e^x \sin 2x \).
   b) \( \frac{\partial u}{\partial t} = 2t \) with \( u(x,0) = e^x \sin 2x \).
   c) \( \frac{\partial u}{\partial t} + u = 0 \) with \( u(x,0) = e^x \sin 2x \).

3. a) If \( u(x,t) \) satisfies \( u_t - 2u_x = 0 \) with \( u(x,0) = 7 + 2 \sin x \), find \( u(x,t) \). [SUGGESTION: Intrepret the differential equation as stating that the directional derivative in a certain direction is zero.]
   b) Find \( v(x,t) \) with the properties \( v_t - 2v_x = 2t \) with \( v(x,0) = 7 + 2 \sin x \).

4. Let \( R \) be an \( n \times n \) matrix with the property that
   \[
   \langle RX, RY \rangle = \langle X, Y \rangle \quad \text{for all} \quad X, Y \in \mathbb{R}^n,
   \]
so \( R \) preserves the inner product and hence the lengths of vectors and the angles between vectors.
   a) Show that \( R^*R = I \), so \( R^* = R^{-1} \).
   b) Conversely, if \( R^* = R^{-1} \), show that \( (1) \) is satisfied.
c) Letting $Y = X$ in equation (11) shows that $R$ preserves the lengths of vectors: 
\[ \|RX\| = \|X\| \] for all $X$. Conversely, if $R$ preserves the lengths of vectors, show that equation (11) is true.

[SUGGESTION: Prove and use the identity 
\[ 4\langle X, Y, = \rangle \|X + Y\|^2 - \|X - Y\|^2 \]

5. [Marsden-Tromba Sec. 3.1 #11] Show that the following functions satisfy the one dimensional wave equation 
\[ u_{tt} = c^2 u_{xx}. \]

a) $u(x, t) = \sin(x - ct)$

b) $u(x, t) = (\sin x)(\sin ct)$

c) $u(x, t) = (x - ct)^6 + (x + ct)^6$.

6. [Marsden-Tromba, Sec 3.1, p.157 #27] A function $u(x, y, z)$ is called harmonic if it satisfies the homogeneous equation 
\[ u_{xx} + u_{yy} + u_{zz} = 0. \]

For which constant(s) $a$ and $b$ is $u(x, y, z) := x^2 + ay^2 + z^2 + byz - 7z$ a harmonic function?

7. [Marsden-Tromba, Sec 3.1, p.158 #31] Show that the Newtonian gravitational potential 
\[ v(x, y, z) := -\frac{GmM}{r} \]

is a harmonic function – except at the origin (see the previous problem). Here $G$, $m$, and $M$ are constants and 
\[ r^2 = x^2 + y^2 + z^2 \]

8. Assume the real-valued functions $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations
\[ u_x = v_y, \]
\[ u_y = -v_x. \]

Show that both $u$ and $v$ are harmonic functions (in two variables), that is they satisfy 
\[ \varphi_{xx} + \varphi_{yy} = 0. \]

9. [Marsden-Tromba, Sec 3.2, p.166 #9] Calculate the second-order Taylor approximation to 
\[ f(x, y) := x \cos \pi y + y \sin \pi x \]
at the point $(1, 2)$.

10. Find all the critical points of $f(x, y) = \sin x \sin y$ and classify them as local maxima, minima, and saddle points.

11. Let $f(x, y) := (x^2 + 4y^2)e^{(1-x^2-y^2)}$. Find and classify all of its critical points. [See the Maple worksheet 
12. [Marsden-Tromba, Sec 3.3, p.184 #46] A smooth function $u(x, y)$ strictly subharmonic if it satisfies $u_{xx} + u_{yy} > 0$. Let $u(x, y)$ be a strictly subharmonic function in the unit disk, $x^2 + y^2 < 1$. Show that $u$ cannot have a local maximum at an interior point of this disk.

**Bonus Problem**

[Please give these directly to Professor Kazdan]

B-1 Let $w(x)$ and $u(x, y)$ be given smooth functions.

a) If $w$ satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function, show that $w$ cannot have a local positive maximum. Also show that $w$ cannot have a local negative minimum.

b) If $u$ satisfies $4u_{xx} + 3u_{yy} - 5u = 0$, show that it cannot have a local positive maximum. Also show that $u$ cannot have a local negative minimum.

c) Repeat the above for a solution of $4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0$.

d) If a function $u(x, y)$ satisfies the above equation in a bounded region $D \subset \mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x, y)$ is zero throughout the region.

[Last revised: February 26, 2012]