Solution to Problem Set 8 #7

Remark: The technique for Problem 11b) is similar. One uses the result from 11a) $u_x = (w_r)(x/r)$ noting that on the right side, $x$ arises in three places: $w_r$, $x$, and $r$.

7. Say $u(x, t)$ is a solution of the one space dimensional wave equation $u_{tt} = u_{xx}$. Find a linear change of variable

$$x = ar + bs, \quad t = cr + ds$$

so that $v(r, s) := u(ar + bs, cr + ds)$ satisfies

$$\frac{\partial v}{\partial r \partial s} = 0.$$

Solution: By the chain rule:

$$v(r, s)_r = u_xx r_r + u_tt r_r = au_x + cu_t. \quad (1)$$

We next want to take the partial derivative of this equation with respect to $s$. This must be done for both $u_x$ and $u_t$. We first compute $\partial u_x(x(r, s), t(r, s))/\partial s$. It may help to let $w(x, t) := u_x(x, t)$. Then, just as in equation (1)

$$\frac{\partial w(x(r, s), t(r, s))}{\partial s} = w_x s + w_t t_s = bw_x + dw_t = bu_{xx} + du_{xt}.$$

Similarly, with $z(x, t) := u_t(x, t)$

$$\frac{\partial z(x(r, s), t(r, s))}{\partial s} = z_x s + z_t t_s = bz_x + dz_t = bu_{tx} + du_{tt}.$$

Consequently,

$$v(r, s)_s = a[bu_{xx} + du_{xt}] + c[bu_{tx} + du_{tt}] = abu_{xx} + (ad + bc)u_{xt} + cd u_{tt}.$$

We would like the right-hand side to be $u_{tt} - u_{xx}$, so

$$ab = -1, \quad ad + bc = 0, \quad cd = 1.$$

One solution is $a = c = d = 1, \quad b = -1$. That is, the desired change of variables is

$$x = r - s, \quad t = r + s.$$