Linear Maps from $\mathbb{R}^2$ to $\mathbb{R}^3$

As an exercise, which I hope you will (soon) realize is entirely routine, we will show that a linear map $F(X) = Y$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ must just be three linear high school equations in two variables:

\begin{align*}
  a_{11}x_1 + a_{12}x_2 &= y_1 \\
  a_{21}x_1 + a_{22}x_2 &= y_2 \\
  a_{31}x_1 + a_{32}x_2 &= y_3
\end{align*}  \tag{1}

Linearity means for any vectors $U$ and $V$ in $\mathbb{R}^2$ and any scalars $c$

\[ F(U + V) = F(U) + F(V) \quad \text{and} \quad F(cU) = cF(U). \]

\textit{Idea:} write $X := (x_2, x_2) \in \mathbb{R}^2$ as

\[ X = x_1(1, 0) + x_2(0, 1) = x_1e_1 + x_2e_2, \quad \text{where} \quad e_1 := (1, 0), \quad e_2 := (0, 1) \]

(physicists often write $e_1$ as $i$ and $e_2$ as $j$ but using this notation in higher dimensions one quickly runs out of letters).

Then, by the two linearity properties

\[ Y = F(X) = F(x_1e_1 + x_2e_2) = F(x_1e_1) + F(x_2e_2) = x_1F(e_1) + x_2F(e_2). \]

But $F(e_1)$ and $F(e_2)$ are just specific vectors in $\mathbb{R}^3$ so this last equation is exactly the desired (1) with

\[ F(e_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \quad \text{and} \quad F(e_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}. \]

Collecting the ingredients we have found that

\[ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = F(x) = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} x_1a_{11} + x_2a_{12} \\ x_1a_{21} + x_2a_{22} \\ x_1a_{31} + x_2a_{32} \end{pmatrix} \]

as claimed in (1).

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