1. Example: quadratic polynomials \( p(x) \) with \( p(1) = 0 \).

2. Polynomial Interpolation.  Find a quadratic polynomial with
\[
p(1) = 1, \quad p(2) = -1, \quad p(4) = 3.
\]

Methods:

**Naive basis:**
\[
p_1(x) := 1, \quad p_2(x) = x, \quad p_3(x) := x^2.
\]
Seek
\[
p(x) = A_1 p_1(x) + A_2 p_2(x) + A_3 p_3(x) = A_1 + A_2 x + A_3 x^2.
\]

**Lagrange’s basis:**
\[
p_1(x) := \frac{(x - 2)(x - 4)}{(1 - 2)(1 - 4)}, \quad p_2(x) := \frac{(x - 1)(x - 4)}{(2 - 1)(2 - 4)}, \quad p_3(x) := \frac{(x - 1)(x - 2)}{(4 - 1)(4 - 2)}.
\]
Note \( p_1(1) = 1 \) while \( p_1(2) = p_1(4) = 0 \), etc. As above, seek
\[
p(x) = A_1 p_1(x) + A_2 p_2(x) + A_3 p_3(x).
\]
Since \( p_1(1) = 1 \) while \( p_2(1) = p_3(1) = 0 \), by setting \( x = 1 \) we immediately find that \( A_1 = 1 \). What are \( A_2 \) and \( A_3 \)?

**Newton’s basis:**
\[
p_1(x) := 1, \quad p_2(x) = (x - 1), \quad p_2(x) := (x - 1)(x - 2)
\]
and seek
\[
p(x) = A_1 p_1(x) + A_2 p_2(x) + A_3 p_3(x).
\]
Letting \( x = 1 \) we find \( A_1 \). Then letting \( x = 2 \) we find \( A_2 \) etc.

**Remark** To evaluate the interpolating polynomial at intermediate points, Newton’s basis uses fewer multiplications. It is also easy to modify it if you want to add an extra interpolating point.

If the data represents the values of some unknown function \( f(x) \) at the specified points, one often would like to understand the error \( \text{Error}(x) := |f(x) - p(x)| \). Newton’s basis is useful for this too.

3. Least Squares What if you have lots of data points
\[
(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)
\]
and want to find the straight line \( y = a + bx \) that best fits the data? In this case trying interpolation you have \( k \) linear equations in the 2 unknowns \( a \) and \( b \):

\[
\begin{align*}
  a + bx_1 &= y_1 \\
  a + bx_2 &= y_2 \\
  \cdots \\
  a + bx_k &= y_k
\end{align*}
\]

Although it is highly unlikely you will find an exact solution, what is the best possible approximation?

**Idea:** Pick \( a \) and \( b \) to minimize the error

\[
E(a, b) := \sqrt{(a + bx_1 - y_1)^2 + (a + bx_1 - y_2)^2 + \cdots + (a + bx_1 - y_k)^2}
\]

4. \( Lu := u'' + u = 0 \).

a) \( \cos x \) and \( \sin x \) for \( x \in \mathbb{R} \) are linearly independent.
   

b) Show that the dimension of the nullspace of \( L \) is at least 2.

c) Show that the dimension of the nullspace of \( L \) is at most 2.

**Moral:** The dimension of the nullspace is exactly 2.

5. \( u'' + u = 2 - x \). There are two tasks:

   - Find the general solution of the homogeneous equation.
   - Find a particular solution of the inhomogeneous equation.


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