DIRECTIONS This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 5 problems (15 points each, so total is 75 points). Maximum score is thus 125 points.
Closed book, no calculators or computers– but you may use one 3" × 5" card with notes on both sides. Clarity and neatness count.

PART A: Five short answer questions (10 points each, so 50 points).

A−1. Which of the following sets are linear spaces? [If not, why not?]
   a) The points $\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0$.
   b) The set of points $(x, y) \in \mathbb{R}^2$ with $y = x^2$.
   c) In $\mathbb{R}^2$, the span of the linearly dependent vectors $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
   d) The set of solutions $\vec{x}$ of $A\vec{x} = 0$, where $A$ is a $4 \times 3$ matrix.
   e) The set of polynomials $p(x)$ of degree at most 2 with $p(1) = 0$.

A−2. Let $S$ be the linear space of $2 \times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$. Find a basis and compute the dimension of $S$.

A−3. Let $S$ and $T$ be linear spaces and $L : S \rightarrow T$ be a linear map. Say $\vec{v}_1$ and $\vec{v}_2$ are (distinct!) solutions of the equations $L\vec{x} = \vec{y}_1$ while $\vec{w}$ is a solution of $L\vec{x} = \vec{y}_2$. Answer the following in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$.
   a) Find some solution of $L\vec{x} = 2\vec{y}_1 - 2\vec{y}_2$.
   b) Find another solution (other than $\vec{w}$) of $L\vec{x} = \vec{y}_2$.

A−4. Say you have a matrix $A$.
   a) If $A : \mathbb{R}^5 \rightarrow \mathbb{R}^5$, what are the possible dimensions of the kernel of $A$? The image of $A$?
   b) If $B : \mathbb{R}^5 \rightarrow \mathbb{R}^3$, what are the possible dimensions of the kernel of $B$? The image of $B$?

A−5. Let $A$ be any $5 \times 3$ matrix so $A\vec{x} : \mathbb{R}^3 \rightarrow \mathbb{R}^5$ is a linear transformation. Answer the following with a brief explanation.
   a) Is $A\vec{x} = \vec{b}$ necessarily solvable for any $\vec{b}$ in $\mathbb{R}^5$?
   b) Suppose the kernel of $A$ is one dimensional. What is the dimension of the image of $A$?

PART B  Five questions, 15 points each (so 75 points total).

B−1. Let $Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. [NOTE: In this problem, there is no partial credit for sloppy computations.]
a) Find the inverse of $Q$.
b) Find the inverse of $Q^2$.

B-2. Define the linear maps $A$, $B$, and $C$ from $\mathbb{R}^2 \to \mathbb{R}^2$ by the rules

- $A$ rotates vectors by $\pi/2$ radians counterclockwise.
- $B$ reflects vectors across the horizontal axis.
- $C$ orthogonal projection onto the vertical axis, so $(x_1, x_2) \to (0, x_2)$

Let $M$ be the linear map that first applies $A$, then $B$, and finally $C$. Find a matrix that represents $M$ in the standard basis for $\mathbb{R}^2$.

B-3. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$ be given matrices.
a) Show that $BA : \mathbb{R}^3 \to \mathbb{R}^3$ cannot be invertible.
b) Give an example where the matrix $AB : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

B-4. a) Find all matrices $A : \mathbb{R}^3 \to \mathbb{R}^2$ whose kernels contain the vector $\vec{x} := \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.
b) Find a basis for the linear space of these matrices.

B-5. Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map that send a polynomial $p(x)$ (of degree at most 2) to $p''(x) + 3p(x)$.
a) Find the matrix representation $[L]_B$ of $L$ using the basis $B = \{1, x, x^2\}$.
b) Find a basis for the kernel of $L$ (you may use your matrix $[L]_B$).
c) Find a basis for the image of $L$ (you may use your matrix $[L]_B$).
d) Is $L$ invertible? Why or why not?