Directions. This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 5 problems (15 points each, so total is 75 points). Maximum score is thus 125 points.

Closed book, no calculators or computers— but you may use one 3” × 5” card with notes on both sides. Clarity and neatness count.

PART A: Five short answer questions (10 points each, so 50 points).

A-1. Which of the following sets are linear spaces? [If not, why not?]

a) The points \( \vec{x} = (x_1, x_2, x_3) \) in \( \mathbb{R}^3 \) with the property \( x_1 - 2x_3 = 0 \).
   Solution: This is a linear space.

b) The set of points \( (x, y) \in \mathbb{R}^2 \) with \( y = x^2 \).
   Solution: This is not a linear space. The point \((1, 1)\) is in this set but \((2, 2)\) is not.

c) In \( \mathbb{R}^2 \), the span of the linearly dependent vectors \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \).
   Solution: This is a linear space. It is the linear space of all points of the form \( (c, -c) \) for any real scalar \( c \). Geometrically, this is a straight line through the origin in the plane \( \mathbb{R}^2 \).

d) The set of solutions \( \vec{x} \) of \( A\vec{x} = 0 \), where \( A \) is a 4 × 3 matrix.
   Solution: This is a linear space: the kernel of \( A \).

e) The set of polynomials \( p(x) \) of degree at most 2 with \( p'(1) = 0 \).
   Solution: This is a linear space since if \( p'(1) = 0 \) and \( q'(1) = 0 \), then both \( (cp) \) and \( (p + q) \) have the same property.

A-2. Let \( S \) be the linear space of \( 2 \times 2 \) matrices \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( a + d = 0 \). Find a basis and compute the dimension of \( S \).
   Solution: Since \( d = -a \), these matrices all have the form
   \[
   \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
   \]
   so the dimension is 3.

A-3. Let \( S \) and \( T \) be linear spaces and \( L : S \to T \) be a linear map. Say \( \vec{v}_1 \) and \( \vec{v}_2 \) are (distinct!) solutions of the equations \( L\vec{x} = \vec{y}_1 \) while \( \vec{w} \) is a solution of \( L\vec{x} = \vec{y}_2 \). Answer the following in terms of \( \vec{v}_1, \vec{v}_2 \), and \( \vec{w} \).

a) Find some solution of \( L\vec{x} = 2\vec{y}_1 - 2\vec{y}_2 \).
   Solution: \( 2\vec{v}_1 - 2\vec{w} \). Another is \( 2\vec{v}_2 - 2\vec{w} \).
b) Find another solution (other than \( \vec{w} \)) of \( L \vec{x} = \vec{y}_2 \).

**Solution:** \( \vec{v}_1 - \vec{v}_2 + \vec{w} \). More generally, \( c(\vec{v}_1 - \vec{v}_2) + \vec{w} \) for any scalar \( c \)

A-4. Say you have a matrix \( A \).

a) If \( A : \mathbb{R}^5 \rightarrow \mathbb{R}^5 \), what are the possible dimensions of the kernel of \( A \)? The image of \( A \)?

**Solution:** 0, 1, \ldots, 5 for both the image and kernel. The special cases \( A = 0 \) and \( A = I \) illustrate the extremes.

b) If \( B : \mathbb{R}^5 \rightarrow \mathbb{R}^3 \), what are the possible dimensions of the kernel of \( B \)? The image of \( B \)?

**Solution:** The image can have dimensions 0, 1, 2, or 3. The kernel can have dimension 2, \ldots, 5.

A-5. Let \( A \) be any \( 5 \times 3 \) matrix so \( A \vec{x} : \mathbb{R}^3 \rightarrow \mathbb{R}^5 \) is a linear transformation. Answer the following with a brief explanation.

a) Is \( A \vec{x} = \vec{b} \) necessarily solvable for any \( \vec{b} \) in \( \mathbb{R}^5 \)?

**Solution:** Since the image can have dimension at most 3 and the target has dimension 5, the map cannot be onto, so there are many vectors \( \vec{b} \) for which \( A \vec{x} = \vec{b} \) has no solution.

b) Suppose the kernel of \( A \) is one dimensional. What is the dimension of the image of \( A \)?

**Solution:** \( \dim(\text{image}(A)) = 3 - 1 = 2 \).

PART B  Five questions, 15 points each (so 75 points total).

B-1. Let \( Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \). [NOTE: In this problem, there is no partial credit for sloppy computations.]

a) Find the inverse of \( Q \).

**Solution:** By a routine computation (the matrix is upper triangular),

\[
Q^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}
\]

b) Find the inverse of \( Q^2 \).

**Solution:** The point of this was that it is simplest to use

\[
Q^{-2} = (Q^{-1})^2 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

B-2. Define the linear maps \( A \), \( B \), and \( C \) from \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by the rules
• A rotates vectors by \(\pi/2\) radians counterclockwise.
• B reflects vectors across the horizontal axis.
• C orthogonal projection onto the vertical axis, so \((x_1, x_2) \rightarrow (0, x_2)\)

Let \(M\) be the linear map that first applies \(A\), then \(B\), and finally \(C\). Find a matrix that represents \(M\) in the standard basis for \(\mathbb{R}^2\).

**Solution:**

**method 1**  
\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{so } M = CBA = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \]

**method 2**  
\[
\begin{align*}
(1,0) & \xrightarrow{A} (0,1) \quad (0,1) & \xrightarrow{A} (-1,0) \\
& \xrightarrow{B} (0,-1) \quad & \xrightarrow{B} (-1,0) \\
& \xrightarrow{C} (0,-1) \quad & \xrightarrow{C} (0,0)
\end{align*}
\]

so \(M = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\)

B-3. Let \(A : \mathbb{R}^3 \to \mathbb{R}^2\) and \(B : \mathbb{R}^2 \to \mathbb{R}^3\) be given matrices.

a) Show that \(BA : \mathbb{R}^3 \to \mathbb{R}^3\) cannot be invertible.

**Solution:**  
Since \(A : \mathbb{R}^3 \to \mathbb{R}^2\), then \(\dim(\ker A) \geq 1\) so there is a \(\vec{x} \in \mathbb{R}^3\), \(\vec{x} \neq 0\) such that \(A\vec{x} = 0\). Consequently \(BA\vec{x} = 0\) so \(BA\) is not one-to-one. Thus it is not invertible.

b) Give an example where the matrix \(AB : \mathbb{R}^2 \to \mathbb{R}^2\) is invertible.

**Solution:**  
Essentially almost any \(A\) and \(B\) will give an example. Perhaps the simplest is  
\[ A : (x_1, x_2, x_3) \rightarrow (x_1, x_2) \quad \text{and} \quad B : (x_1, x_2) \rightarrow (x_1, x_2, 0). \]

As matrices, \(A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\) and \(B := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\).

B-4. a) Find all matrices \(A : \mathbb{R}^3 \to \mathbb{R}^2\) whose kernels contain the vector \(\vec{x} := \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\).

**Solution:**  
Say \(A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\). Then \[0 = A\vec{x} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} a - 2c \\ d - 2f \end{pmatrix}\]  
so \(a = 2c\) and \(d = 2f\). Thus \[A = \begin{pmatrix} 2c & b & c \\ 2f & e & f \end{pmatrix}\]

for any scalars \(b, c, e,\) and \(f\).
b) Find a basis for the linear space of these matrices.

**SOLUTION:** From the previous part

\[ A = c \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} + e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

so the 4 matrices above are a basis.

B-5. Let \( L : \mathcal{P}_2 \to \mathcal{P}_2 \) be the linear map that send a polynomial \( p(x) \) (of degree at most 2) to \( p''(x) + 3p(x) \).

a) Find the matrix representation \([L]_B\) of \( L \) using the basis \( B = \{1, x, x^2\} \).

**SOLUTION:** By a straightforward computation, if \( p(x) = a + bx + cx^2 \), then

\[ Lp(x) = 2c + 3(a + bx + cx^2) = (3a + 2c)1 + 3bx + 3cx^2. \]

If \( q(x) = \alpha + \beta x + \gamma x^2 \), we can seek a polynomial \( p(x) \in \mathcal{P}_2 \) so that \( Lp = q \). Comparing \( Lp \) and \( q \) above, we need to pick the coefficients \( a \), \( b \), and \( c \) so that

\[
\begin{align*}
3a + 2c &= \alpha \\
3b &= \beta \\
3c &= \gamma
\end{align*}
\]

that is,

\[
\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.
\]

(1)

The 3 \( \times \) 3 matrix above is the desired matrix \([L]_B\) of \( L \) in the basis \( B = \{1, x, x^2\} \).

Let’s do this more formally. First we explicitly introduce the basis

\[
e_1(x) := 1, \quad e_2(x) := x, \quad e_3(x) = x^2.
\]

Then

\[ p(x) = ae_1(x) + be_2(x) + ce_3(x) \]

and

\[ Le_1 = 3 = 3e_1, \quad Le_2 = 3x = 3e_2, \quad Le_3 = 2 + 3x^2 = 2e_1 + 3e_3. \]

This gives the 3 columns of the matrix on the right in (1)

b) Find a basis for the kernel of \( L \) (you may use your matrix \([L]_B\)).

**SOLUTION:** Either solve the there equations

\[
\begin{align*}
3a + 2c &= 0 \\
3b &= 0 \\
3c &= 0
\end{align*}
\]

or find the kernel of the matrix \( \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \). Both instantly show that the kernel of \( L \) acting on quadratic polynomials only has \( p(x) = 0 \).
c) Find a basis for the image of $L$ (you may use your matrix $[L]_B$).

**Solution:** Since $[L]_B$ is a square matrix whose kernel is trivial (only the vector representing the polynomial $p(x) \equiv 0$), it is invertible, so any basis for $\mathbb{P}_2$, such as $\{1, x, x^2\}$ is a basis for the image.

d) Is $L$ invertible? Why or why not?

**Solution:** It is invertible. See the answer in part c).