Fourier Series of \( f(x) = x \)

Given a real periodic function \( f(x) \), \(-\pi < x < \pi\), one can find its Fourier series in two (equivalent) ways: using trigonometric functions:

\[
f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \left[ a_k \frac{\cos kx}{\sqrt{\pi}} + b_k \frac{\sin kx}{\sqrt{\pi}} \right]
\]

or using the complex exponential

\[
f(x) = \sum_{k=-\infty}^{\infty} c_k \frac{e^{ikx}}{\sqrt{2\pi}}.
\]

Note that if \( f(x) \) is a real-valued function, we can take the real part of the complex exponential version to get the trigonometric version (caution: the coefficients \( c_k \) will probably be complex numbers).

Here we will use complex exponentials. The Fourier coefficients are

\[
c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} xe^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x[\cos kx - i\sin kx] \, dx = \frac{-2i}{\sqrt{2\pi}} \int_{0}^{\pi} x\sin kx \, dx
\]

But

\[
\int_{0}^{\pi} x\sin kx \, dx = -\frac{x \cos kx}{k} \bigg|_{0}^{\pi} + \frac{1}{k} \int_{0}^{\pi} \cos kx \, dx = -\frac{\pi \cos k\pi}{k} = -\frac{\pi}{k} (-1)^k.
\]

Thus

\[
c_k = -\frac{2i}{\sqrt{2\pi}} \left[ -\frac{\pi}{k} (-1)^k \right] = i\sqrt{2\pi} \left[ \frac{(-1)^k}{k} \right].
\]

Consequently

\[
x = i\sqrt{2\pi} \sum_{k \neq 0} \frac{(-1)^k}{k} \frac{e^{ikx}}{\sqrt{2\pi}} = i \sum_{k \neq 0} \frac{(-1)^k}{k} e^{ikx}
\]

\[
= -2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin kx = 2 \left[ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \right]
\]

Finally we compute what the Pythagorean Theorem tells us: \( ||x||^2 = \sum |c_k|^2 \). Since

\[
||x||^2 = \int_{-\pi}^{\pi} |x|^2 \, dx = \frac{2}{3} \pi^3,
\]

and

\[
\sum |c_k|^2 = 2\pi \left[ -\sum_{-\infty}^{-1} \frac{1}{k^2} + \sum_{1}^{\infty} \frac{1}{k^2} \right] = 4\pi \sum_{1}^{\infty} \frac{1}{k^2}
\]

Therefore

\[
\frac{2}{3} \pi^3 = 4\pi \sum_{1}^{\infty} \frac{1}{k^2}, \quad \text{that is,} \quad \sum_{1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.
\]

Interesting! – and not obvious at all.