Functions of one variable (review).

Interpret the function:
   as a **graph** $y = f(x)$
   as the **position of a particle** $y = g(t)$ at time $t$.

The derivative: **slope of tangent line** or **velocity**.

At a local maximum or minimum the derivative is zero.
**Example:** Standard minimum \( f(x, y) = x^2 + 3y^2 \)

Find critical points:

\[
\partial_x f(x, y) = 2x, \quad \partial_y f(x, y) = 6y
\]

so the only critical point is the origin, \((0, 0)\).

Second derivative test:

\[
\partial_{xx} f(x, y) = 2, \quad \partial_{xy} f(x, y) = 0, \quad \partial_{yy} f(x, y) = 6
\]

\( f''(0, 0) \) is the diagonal matrix

\[
f''(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.
\]

This is *positive definite* so the origin is a local minimum.
**Example:** Standard maximum \( f(x, y) = -(x^2 + y^2) \)

![Graph of standard maximum function](image1.png)

**Example:** Standard saddle \( f(x, y) = -x^2 + 3y^2 \)

![Graph of standard saddle function](image2.png)
**Example:** \[ f(x, y) = \frac{3x^4 - 4x^3 - 12x^2 + 12}{12(1 + y^2)} \]

The curve on the left is \( f(x, 0) \). From the graph you see one saddle, one max, and one min, all on the \( x \) axis.

Compute the critical points:

\[ \partial_x f(x, y) = \frac{x^3 - x^2 - 2x}{1 + y^2}, \quad \partial_y f(x, y) = \frac{- (3x^4 - 4x^3 - 12x^2 + 12)y}{6(1 + y^2)^2} \]

Critical points: \((0, 0),\ (−1, 0),\ (2, 0)\).
Second derivative test. The second partial derivatives take more work to compute:

\[
f''(x, y) = \begin{pmatrix}
\frac{3x^2 - 2x - 2}{1+y^2} & -\frac{2(3x^3 - x^2 - 2x)y}{(1+y^2)^2} \\
-\frac{2(x^3 - x^2 - 2x)y}{(1+y^2)^2} & \frac{-2(x^3 - x^2 - 2x)y}{(1+y^2)^2}
\end{pmatrix}
\]

Thus, the second derivative matrices at the critical points are:

\[
f''(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{max}
\]

\[
f''(2, 0) = \begin{pmatrix} 6 & 0 \\ 0 & \frac{10}{3} \end{pmatrix} \quad \text{min}
\]

\[
f''(-1, 0) = \begin{pmatrix} 3 & 0 \\ 0 & \frac{-7}{6} \end{pmatrix} \quad \text{saddle}
\]
EXAMPLES OF DEGENERATE CRITICAL POINTS
Moral: the second derivative test is inconclusive.

Degenerate saddle at the origin:

\[ f(x, y) = x^2 + y^3 \]

\[ f''(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \]
Degenerate minimum at the origin:

\[ f(x,y) = x^2 + y^4 \]

\[ f''(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \]
Degenerate maximum at the origin:

\[ f(x, y) = -(x^4 + y^4) \]

\[ f''(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]
Degenerate *monkey* saddle at the origin:

\[ f(x, y) = x^3 - 3xy^2 = \Re\{(x + iy)^3\} \]

\[ f''(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]
**Example:** \( f(x, y) = (2x^2 + 3y^2)e^{(1-x^2-y^2)} \)

Clearly we see five critical points: two maxima, two saddle points, and one minima (in the pit).

Find them:

\[
\partial_x f(x, y) = 2x[2 - (2x^2 + 3y^2)]e^{(1-x^2-y^2)}
\]
\[
\partial_y f(x, y) = 2y(3 - (2x^2 + 3y^2)]e^{(1-x^2-y^2)}.
\]

So \( \partial_x f(x, y) = 0 \) and \( \partial_y f(x, y) = 0 \) at the five points

\((0, 0), \ (\pm 1, 0), \ \text{and} \ (0, \pm 1)\).
Classify the critical points (second derivative test):

\[
\begin{align*}
\partial_{xx} f(x, y) &= 2[2 - 8x^2 - (1 - 2x^2)(2x^2 + 3y^2)]e^{1-x^2-y^2} \\
\partial_{xy} f(x, y) &= 4xy[-5 + (2x^2 + 3y^2)]e^{1-x^2-y^2} \\
\partial_{yy} f(x, y) &= 2[3 - 12y^2 - (1 - 2y^2)(2x^2 + 3y^2)]e^{1-x^2-y^2}
\end{align*}
\]

Thus the second derivative (Hessian) matrices

\[
f''(x, y) = \begin{pmatrix}
\partial_{xx} f(x, y) & \partial_{xy} f(x, y) \\
\partial_{xy} f(x, y) & \partial_{yy} f(x, y)
\end{pmatrix}
\]

at these five critical points are (as anticipated)

\[
\begin{align*}
f''(0, 0) &= \begin{pmatrix} 4e & 0 \\ 0 & 6e \end{pmatrix} \quad \text{local minimum} \\
f''(\pm 1, 0) &= \begin{pmatrix} -8 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{saddles} \\
f''(0, \pm 1) &= \begin{pmatrix} -2 & 0 \\ 0 & -12 \end{pmatrix} \quad \text{maxima.}
\end{align*}
\]

**Exercise**  Let \( A \) be an \( n \times n \) real invertible symmetric matrix and \( f(X) := \langle x, Ax \rangle e^{-\|X\|^2}, \ X \in \mathbb{R}^n \). Show that critical points of \( f \) are precisely the origin and the \( \pm \) unit eigenvectors of \( A \). If the eigenvalues of \( A \) are distinct, there are \( 2n+1 \) critical points. [The classification of these critical points is more complicated – but reasonable. For instance, it is clear that \( f''(0) = 2A \).]
"Intuition" is Unreliable

Let $f(x,y)$ be a smooth function on $\mathbb{R}^2$ with only one critical point: a strict local minimum at the origin.

Must this be the global minimum?

For a function of one variable, this must be the global min – but not for functions of several variables. The simplest example is probably the polynomial

$$f(x,y) := (1 - y)^3 x^2 + y^2$$

Perhaps easier to visualize are

$$f(x,y) := (1 - y^2)^3 x^2 + y^2 \quad \text{and} \quad g(x,y) := \frac{(1 - y^2)^3 x^2 + y^2}{(1 + y^2)^3}$$