Problem Set 10
Due: Never.

1. [Bretscher, Sec. 7.1 #12] Find the eigenvalues and eigenvectors of $A := \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$.

2. [Bretscher, p. 318, Sec. 7.2 #28] [Problem on Markov Chains].

3. If $\vec{v}$ is an eigenvector of the matrix $A$, show that it is also an eigenvector of $A + 37I$. What is the corresponding eigenvalue?

4. Let $A$ be an invertible matrix. Show that $\lambda = 0$ cannot be an eigenvalue.
   Conversely, if a (square) matrix is not invertible, show that $\lambda = 0$ is an eigenvalue.

5. Let $z = x + iy$ be a complex number. For which real numbers $x$, $y$ is $|e^z| < 1$?

6. Let $M$ be a $4 \times 4$ matrix of real numbers. If you know that both $1 + 2i$ and $2 - i$ are eigenvalues of $M$, is $M$ diagonalizable? Proof or counterexample.

7. Let $A$ and $B$ be $n \times n$ real positive definite matrices and let $C := tA + (1 - t)B$. If $0 \leq t \leq 1$, show that $C$ is also positive definite. [This is simple. No “theorems” are needed.]

8. Let $A := \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   a) Find the eigenvalues and eigenvectors of $A$.
   b) Find an orthogonal transformation $R$ so that $R^{-1}AR$ is a diagonal matrix.

9. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, solve $\frac{d\vec{x}}{dt} = A\vec{x}$ with initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

10. Let $A$ and $B$ be any $3 \times 3$ matrices. Show that $\text{trace}(AB) = \text{trace}(BA)$. [This is also true for $n \times n$ matrices.]
    Use this to give another proof that if the matrices $M$ and $Q$ are similar, then $\text{trace}(M) = \text{trace}(Q)$.
11. Let \( A := \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} \).

a) Compute \( A^{50} \).

b) Let \( P_0 := \begin{pmatrix} p \\ q \end{pmatrix} \) where \( p > 0 \) and \( q > 0 \) with \( p + q = 1 \). Compute \( A^{50}P_0 \). What do you suspect \( \lim_{k \to \infty} A^kP_0 = ? \).

c) Note that \( A \) is the transition matrix of a Markov process. What do you suspect is the long-term stable state? Verify your suspicion.

12. Let \( A \) be a 3 \( \times \) 3 matrix whose eigenvalues are \(-1 \pm i\) and \(-2\). If \( \vec{x}(t) \) is a solution of \( \frac{d\vec{x}}{dt} = A\vec{x} \), show that \( \lim_{t \to \infty} \vec{x}(t) = 0 \) independent of the initial value \( \vec{x}(0) \).

13. a) If \( B := \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \), find a self adjoint matrix \( Q \) so that \( Q^2 = B \). [This should be obvious.]

b) If \( A := \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \), find a self adjoint matrix \( P \) so that \( P^2 = A \).

14. Let \( A := \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \). Solve \( \frac{d^2\vec{x}(t)}{dt^2} + A\vec{x}(t) = 0 \) with \( \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \vec{x}'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

[Remark: If \( A \) were the diagonal matrix \( \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \), then this problem would have been simple.]

15. Let \( A \) be an \( n \times n \) matrix that commutes with all \( n \times n \) matrices, so \( AB = BA \) for all matrices \( B \). Show that \( A = cI \) for some scalar \( c \). [Suggestion: Let \( \vec{v} \) be an eigenvector of \( A \) with eigenvalue \( \lambda \)].

[Last revised: December 3, 2012]