1. [Bretscher, Sec. 5.1 #16] Consider the following vectors in $\mathbb{R}^4$

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

Can you find a vector $\vec{u}_4$ in $\mathbb{R}^4$ such that the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ are orthonormal? If so, how many such vectors are there?

2. [Bretscher, Sec. 5.1 #17] Find a basis for $W^\perp$, where

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right\}.$$

3. [Bretscher, Sec. 5.1 #21] Find scalars $a, b, c, d, e, f, g$ so that the following vectors are orthonormal:

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \quad \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \quad \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}.$$

4. [Bretscher, Sec. 5.1 #26] Find the orthogonal projection $P_S$ of $\vec{x} := \begin{pmatrix} 49 \\ 49 \end{pmatrix}$ into the subspace $S$ of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} -3 \\ -6 \\ 2 \end{pmatrix}$.

5. [Bretscher, Sec. 5.1 #37] Consider a plane $V$ in $\mathbb{R}^3$ with orthonormal basis $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x}$ be a vector in $\mathbb{R}^3$. Find a formula for the reflection $R\vec{x}$ of $\vec{x}$ across the plane $V$. 

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Problem Set 5

Due: In class Thursday, Oct. 18 Late papers will be accepted until 1:00 PM Friday.

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple mental exercises. These are not to be handed in.

Sec. 5.1, #28, 29, 31
Sec. 5.2 #33

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1. [Bretscher, Sec. 5.1 #16] Consider the following vectors in $\mathbb{R}^4$

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

Can you find a vector $\vec{u}_4$ in $\mathbb{R}^4$ such that the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ are orthonormal? If so, how many such vectors are there?

2. [Bretscher, Sec. 5.1 #17] Find a basis for $W^\perp$, where

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right\}.$$

3. [Bretscher, Sec. 5.1 #21] Find scalars $a, b, c, d, e, f, g$ so that the following vectors are orthonormal:

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \quad \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \quad \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}.$$

4. [Bretscher, Sec. 5.1 #26] Find the orthogonal projection $P_S$ of $\vec{x} := \begin{pmatrix} 49 \\ 49 \end{pmatrix}$ into the subspace $S$ of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} -3 \\ -6 \\ 2 \end{pmatrix}$.

5. [Bretscher, Sec. 5.1 #37] Consider a plane $V$ in $\mathbb{R}^3$ with orthonormal basis $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x}$ be a vector in $\mathbb{R}^3$. Find a formula for the reflection $R\vec{x}$ of $\vec{x}$ across the plane $V$. 

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6. [Bretscher, Sec. 5.2 #32] Find an orthonormal basis for the plane \( x_1 + x_2 + x_3 = 0 \).

7. Let \( V \) be a linear space. A linear map \( P : V \to V \) is called a projection if \( P^2 = P \) (this \( P \) is not necessarily an “orthogonal projection”).

a) Show that the matrix \( P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \) is a projection. Draw a sketch of \( \mathbb{R}^2 \) showing the vectors \((1,2), (-1,0), (3,1), \text{and} (0,3) \) and their images under the map \( P \). Also indicate both the image, \( V \), and kernel, \( W \), of \( P \).

b) Repeat this for the complementary projection \( Q := I - P \).

c) If the image and kernel of a projection \( P \) are orthogonal then \( P \) is called an orthogonal projection. [This of course now assumes that \( V \) has an inner product.]

Let \( M = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \). For which real value(s) of \( a \) and \( c \) is this a projection? An orthogonal projection?

The remaining problems are from the Lecture notes on Vectors


8. [p. 8 #5] The origin and the vectors \( X, Y, \text{and} X + Y \) define a parallelogram whose diagonals have length \( X + Y \) and \( X - Y \). Prove the parallelogram law

\[
\| X + Y \|^2 + \| X - Y \|^2 = 2 \| X \|^2 + 2 \| Y \|^2 ;
\]

This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

9. [p. 8 #6]

a) Find the distance from the straight line \( 3x - 4y = 10 \) to the origin.

b) Find the distance from the plane \( ax + by + cz = d \) to the origin (assume the vector \( \vec{N} = (a, b, c) \neq 0 \)).

10. [p. 8 #8]

a) If \( X \) and \( Y \) are real vectors, show that

\[
\langle X, Y \rangle = \frac{1}{4} \left( \| X + Y \|^2 - \| X - Y \|^2 \right).
\]

This formula is the simplest way to recover properties of the inner product from the norm.

b) As an application, show that if a square matrix \( R \) has the property that it preserves length, so \( \| RX \| = \| X \| \) for every vector \( X \), then it preserves the inner product, that is, \( \langle RX, RY \rangle = \langle X, Y \rangle \) for all vectors \( X \) and \( Y \).
11. [p. 9 #10]
   a) If a certain matrix $C$ satisfies $\langle X, CY \rangle = 0$ for all vectors $X$ and $Y$, show that $C = 0$.
   b) If the matrices $A$ and $B$ satisfy $\langle X, AY \rangle = \langle X, BY \rangle$ for all vectors $X$ and $Y$, show that $A = B$.

   a) Give an example of a $3 \times 3$ anti-symmetric matrix.
   b) If $A$ is any anti-symmetric matrix, show that $\langle X, AX \rangle = 0$ for all vectors $X$.
   c) Say $X(t)$ is a solution of the differential equation $\frac{dX}{dt} = AX$, where $A$ is an anti-symmetric matrix. Show that $\|X(t)\| = \text{constant}$. [Remark: A special case is that $X(t) := \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ satisfies $X' = AX$ with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so this problem gives another proof that $\cos^2 t + \sin^2 t = 1$.]

**Bonus Problem**

[Please give this directly to Professor Kazdan]

1-B This is a followup to problem 7.
   a) If a projection $P$ is self-adjoint, so $P^* = P$, show that $P$ is an orthogonal projection.
   b) Conversely, if $P$ is an orthogonal projection, show that it is self-adjoint.

[Last revised: October 22, 2012]