Problem Set 6

Due: In class Thursday, Oct. 25 Late papers will be accepted until 1:00 PM Friday.

Remark: We have completed Chapter 5, Sections 5.1, 5.2, 5.3, and 5.4 (except for the QR Factorization – which we will skip). Since Fall Break interrupts this week, this assignment will be shorter.

1. [Bretscher, Sec. 5.2 #34] Find an orthonormal basis for the kernel of the matrix

\[ A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}. \]

2. [Bretscher, Sec. 5.4 #20] Using pencil and paper, find the least-squares solution to \( A \vec{x} = \vec{b} \) where

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}. \]

3. Use the Method of Least Squares to find the parabola \( y = ax^2 + b \) that best fits the following data given by the following four points \((x_j, y_j), j = 1, \ldots, 4:\)

\((-2, 4), \quad (-1, 3), \quad (0, 1), \quad (2, 0).\)

Ideally, you’d like to pick the coefficients \(a\) and \(b\) so that the four equations \(ax_j^2 + b = y_j, j = 1, \ldots, 4\) are all satisfied. Since this probably can’t be done, one uses least squares to find the best possible \(a\) and \(b\).

4. The water level in the North Sea is mainly determined by the so-called M2 tide, whose period is about 12 hours. The height \( H(t) \) thus roughly has the form

\[ H(t) = c + a \sin(2\pi t/12) + b \cos(2\pi t/12), \]

where time \( t \) is measured in hours (note \( \sin(2\pi t/12) \) and \( \cos(2\pi t/12) \) are periodic with period 12 hours). Say one has the following measurements:

\[
\begin{array}{c|ccccccc}
 t \ (\text{hours}) & 0 & 2 & 4 & 6 & 8 & 10 \\
 H(t) \ (\text{meters}) & 1.0 & 1.6 & 1.4 & 0.6 & 0.2 & 0.8 \\
\end{array}
\]

Use the method of least squares to find the constants \(a\), \(b\), and \(c\) in \(H(t)\) for this data.

5. Let \( A \) be a real matrix, not necessarily square.

a) Show that both \( A^*A \) and \( AA^* \) are self-adjoint.

b) Show that \( \ker A = \ker A^*A \). [Hint: Show separately that \( \ker A \subseteq \ker A^*A \) and \( \ker A \supseteq \ker A^*A \). The identity \( \langle \vec{x}, A^*A\vec{x} \rangle = \langle A\vec{x}, A\vec{x} \rangle \) is useful.]

[Last revised: October 25, 2012]