Problem Set 8

Due: In class Thursday, Nov. 8 Late papers will be accepted until 1:00 PM Friday.

Some of this is on the material in Bretscher, Sec. 5.5, concerning inner products in spaces of functions. No new ideas are involved, but it does take time to simply relax.

1. For a square matrix $A$, a scalar $\lambda$ is an eigenvalue and a vector $v \neq 0$ is a corresponding eigenvector if $Av = \lambda v$, so $A$ maps $v$ to a multiple of itself.

If $A$ is a symmetric (that is, self-adjoint) matrix with eigenvalues $\lambda, \mu, \lambda \neq \mu$ and corresponding eigenvectors $v$ and $w$. Show that $v$ and $w$ are orthogonal.

2. Introduce the following inner product on the space of continuous functions on the interval $-1 \leq x \leq 1$: $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

   a) Show that $1 \perp x$.
   b) For which constants $a, b$ is $f(x) := a + bx + x^2$ orthogonal to both $1$ and $x$?
   c) Find an orthogonal basis for the span of $1$, $x$, and $x^2$.

3. A real-valued function is called even if $f(-x) = f(x)$ for all $x$, and odd if $f(-x) = -f(x)$ for all $x$. For instance, $2x^4 + x\sin 3x$ is even and $\sin 4x - 7x^5$ is odd. Using the same inner product as above,

   a) Show that any odd function $f(x)$ is orthogonal to the function $1$.
   b) Show that any even function $f(x)$ is orthogonal to $\sin 13x$.
   c) Show that the product of an even function $f(x)$ and an odd function $g(x)$ is odd.
   d) Show that any even function $f(x)$ is orthogonal to any odd function $g(x)$.

4. [Bretscher, Sec. 5.5 #16] Consider the space of continuous functions on the interval $[0, 1]$ (that is, $0 \leq x \leq 1$) with the inner product $\langle f, g \rangle := \int_{0}^{1} f(x)g(x)\,dx$.

   a) Using this inner product, find an orthonormal basis for the space $P_1$ of polynomials of degree at most one.
   b) Find a linear polynomial $g(x) = a + bx$ that best approximates $x^2$ in the norm defined by this inner product.

5. [Bretscher, Sec. 5.5 #20]. In $\mathbb{R}^2$ consider the NEW inner product $\langle v, w \rangle := v^T \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} w$ with corresponding norm $\|v\|^2 := \langle v, v \rangle$.

   a) Find all vectors in $\mathbb{R}^2$ that are orthogonal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
   b) Find an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product.
6. [Bretscher, Sec. 5.5 #24]. Using the inner product of problem 4, for the polynomials $f$, $g$, and $h$ say we are given the following table of inner products:

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$g$</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$h$</td>
<td>8</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

For example, $\langle g, h \rangle = \langle h, g \rangle = 3$. Let $E$ be the span of $f$ and $g$.

a) Compute $\langle f, g + h \rangle$.

b) Compute $\|g + h\|$.

c) Find $\text{proj}_E h$. [Express your solution as linear combinations of $f$ and $g$.]

d) Find an orthonormal basis of the span of $f$, $g$, and $h$ [Express your results as linear combinations of $f$, $g$, and $h$.]

7. [Like Bretscher, Sec. 5.5 #26 & 28]. Use the inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

Define

$$f(x) = \begin{cases} 
-1 & \text{if } -\pi < x \leq 0, \\
1 & \text{if } 0 < x \leq \pi,
\end{cases}$$

and extend $f$ to all of $\mathbb{R}$ as period is with period $2\pi$: $f(x + 2\pi) = f(x)$. This is called a square wave.

a) Compute the first $N$ terms in the Fourier Series

$$f(x) = A_0 + \sum_{k=1}^{N} [A_k \cos kx + B_k \sin kx]$$

b) Apply the Pythagorean Theorem 5.5.6 (page 343) to your answer.

8. Compute the determinant of the upper triangular matrix

$$A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & 0 & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix}$$

[Do the cases $n = 2$ and $n = 3$ first.]

9. The $n \times n$ matrices $A$ and $B$ are similar if there is and invertible $n \times n$ matrix $S$ so that $B = SAS^{-1}$. If $A$ and $B$ are similar, show that $\det B = \det A$.

[Last revised: November 14, 2012]