Problem Set 8
DUE: In class Thursday, Nov. 8 Late papers will be accepted until 1:00 PM Friday.

Some of this is on the material in Bretscher, Sec. 5.5, concerning inner products in spaces of functions. No new ideas are involved, but it does take time to simply relax.

1. For a square matrix $A$, a scalar $\lambda$ is an eigenvalue and a vector $\vec{v} \neq 0$ is a corresponding eigenvector if $A\vec{v} = \lambda\vec{v}$, so $A$ maps $\vec{v}$ to a multiple of itself.

If $A$ is a symmetric (that is, self-adjoint) matrix with eigenvalues $\lambda, \mu, \lambda \neq \mu$ and corresponding eigenvectors $\vec{v}$ and $\vec{w}$. Show that $\vec{v}$ and $\vec{w}$ are orthogonal.

**Solution:** We know that $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$, where $\mu \neq \lambda$. Using the inner product we have

$$\langle A\vec{v}, \vec{w} \rangle = \lambda \langle \vec{v}, \vec{w} \rangle \quad \text{and} \quad \langle A\vec{w}, \vec{v} \rangle = \mu \langle \vec{w}, \vec{v} \rangle.$$ 

But since $A = A^*$, then $\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A\vec{w} \rangle \langle A\vec{w}, \vec{v} \rangle$. Consequently,

$$\lambda \langle \vec{v}, \vec{w} \rangle = \mu \langle \vec{w}, \vec{v} \rangle.$$

Since $\lambda \neq \mu$, then $\langle \vec{v}, \vec{w} \rangle = 0$.

2. Introduce the following inner product on the space of continuous functions on the interval $-1 \leq x \leq 1$: $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx$.

a) Show that $1 \perp x$.

**Solution:** $\langle 1, x \rangle = \int_{-1}^{1} 1 \cdot x \, dx = \frac{1}{2} x^2 \bigg|_{-1}^{1} = 0$.

b) For which constants $a, b$ is $f(x) := ax + bx + x^2$ orthogonal to both 1 and $x$?

**Solution:** We want $\langle a + bx + x^2, 1 \rangle = 0$ and $\langle a + bx + x^2, x \rangle = 0$, that is

$$0 = \int_{-1}^{1} (a + bx + x^2) \, dx = 2a + 1 \quad \text{and} \quad 0 = \int_{-1}^{1} (a + bx + x^2) x \, dx = b,$$

so $a = -1/2$ and $b = 0$.

c) Find an orthogonal basis for the span of 1, $x$, and $x^2$.

**Solution:** $e_1(x) = 1, \ e_2(x) = x, \ e_3(x) = -\frac{1}{2} + x^2$.

3. A real-valued function is called even if $f(-x) = f(x)$ for all $x$, and odd if $f(-x) = -f(x)$ for all $x$. For instance, $2x^4 + x \sin 3x$ is even and $\sin 4x - 7x^5$ is odd. Using the same inner product as above,
a) Show that any odd function \( f(x) \) is orthogonal to the function 1.

**SOLUTION:** Since \( f(x) \) is odd, using the substitution \( t = x \) in the second step
\[
\langle f, 1 \rangle = \int_{-1}^{1} f(x) \cdot 1 \, dx = \int_{-1}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx \\
= \int_{0}^{1} f(-t) \, dt + \int_{0}^{1} f(x) \, dx \\
= -\int_{0}^{1} f(t) \, dt + \int_{0}^{1} f(x) \, dx = 0.
\]

b) Show that any even function is orthogonal to \( \sin 13x \).

**SOLUTION:** Almost identical to part (a).

c) Show that the product of an even function \( f(x) \) and an odd function \( g(x) \) is odd.

**SOLUTION:** Let \( h(x) = f(x)g(x) \). Then
\[
h(-x) = f(-x)g(-x) = -f(x)g(x) = -h(x).
\]

d) Show that any even function \( f(x) \) is orthogonal to any odd function \( g(x) \).

**SOLUTION:** Let \( h(x) = f(x)g(x) \). Since \( h \) is odd, this follows from part (a).

4. [Bretscher, Sec. 5.5 #16] Consider the space of continuous functions on the interval \([0, 1]\) (that is, \(0 \leq x \leq 1\)) with the inner product \( \langle f, g \rangle := \int_{0}^{1} f(x)g(x) \, dx \).

a) Using this inner product, find an orthonormal basis for the space \( P_1 \) of polynomials of degree at most one.

**SOLUTION:** We use the Gram-Schmidt process. Pick the constant \( c \) so that \( x = x \cdot 1 + w \), where \( w(x) \perp 1 \). Then
\[
\langle x, 1 \rangle = c\langle 1, 1 \rangle + \langle w, 1 \rangle = c \int_{0}^{1} 1^2 \, dx = c + 0 = c.
\]

Since \( \langle x, 1 \rangle = \int_{0}^{1} x \, dx = 1/2 \), then \( c = 1/2 \). Consequently \( w(x) = x - \frac{1}{2} \) is orthogonal to the function 1. Therefore \( v_1(x) = 1 \) and \( v_2(x) = x - \frac{1}{2} \) is an orthogonal basis.

Because \( \|v_1\|^2 = \int_{0}^{1} 1^2 \, dx = 1 \) and \( \|v_2\|^2 = \int_{0}^{1} (x - \frac{1}{2})^2 \, dx = \frac{1}{12} \), an orthonormal basis is \( e_1(x) = 1 \) and \( e_2(x) = \sqrt{\frac{12}{3}}(x - \frac{1}{2}) \).

b) Find a linear polynomial \( g(x) = a + bx \) that best approximates \( x^2 \) in the norm defined by this inner product.

**SOLUTION:** The best approximation (in this norm) of \( f(x) = x^2 \) by a function of the form \( a + bx \) is the orthogonal projection of \( x^2 \) into this space. We thus seek constants \( \alpha \) and \( \beta \) so that
\[
x^2 = \alpha e_1(x) + \beta e_2(x) + w(x),
\]
where \( w(x) \) is orthogonal to 1 and \( x \), or equivalently, to \( e_1 \) and \( e_2 \). To find \( \alpha \) and \( \beta \), as usual we take the inner product of (1) with \( e_1 \) and \( e_2 \) to find

\[
\alpha = \langle x^2, e_1 \rangle = \int_0^1 x^2 \cdot 1 \, dx = \frac{1}{3} \quad \text{and} \quad \beta = \langle x^2, e_2 \rangle = \sqrt{\frac{12}{12}} \int_0^1 x^2 \left( x - \frac{1}{2} \right) = \frac{1}{\sqrt{12}}.
\]

Thus the best approximation (using this inner product) is \( \frac{1}{3} + (x - \frac{1}{2}) \).

5. [Bretscher, Sec. 5.5 #20]. In \( \mathbb{R}^2 \) consider the inner product \( \langle \vec{v}, \vec{w} \rangle := \vec{v}^T \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \vec{w} \) with corresponding norm \( |||\vec{v}|||^2 := \langle \vec{v}, \vec{v} \rangle \).

a) Find all vectors in \( \mathbb{R}^2 \) that are orthogonal to \( \vec{v} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

**Solution:** The condition \( \vec{w} := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \) being orthogonal to \( \vec{v} \) means

\[
0 = \langle \vec{v}, \vec{w} \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1 + 2w_2,
\]

so \( \vec{w} = c \begin{pmatrix} -2 \\ 1 \end{pmatrix} \) where \( c \) is any scalar.

b) Find an orthonormal basis for \( \mathbb{R}^2 \) with respect to this inner product.

**Solution:** The vectors \( \vec{v} \) and \( \vec{w} \) are an orthogonal basis. To get an orthonormal basis we just make them into unit vectors – using the norm associated with this new inner product \( |||\vec{v}|||^2 := \langle \vec{v}, \vec{v} \rangle = \vec{v}^T \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \vec{v} \). Then \( |||\vec{v}|||^2 = 1 \) and \( |||\vec{w}|||^2 = 4c^2 \) so one orthonormal basis is

\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.
\]

6. [Bretscher, Sec. 5.5 #24]. Using the inner product of problem 4, for the polynomials \( f \), \( g \), and \( h \) say we are given the following table of inner products:

<table>
<thead>
<tr>
<th>\langle \cdot, \cdot \rangle</th>
<th>f</th>
<th>g</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>g</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>h</td>
<td>8</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

For example, \( \langle g, h \rangle = \langle h, g \rangle = 3 \). Let \( E \) be the span of \( f \) and \( g \).
a) Compute \( \langle f, g + h \rangle \).

**Solution:** \( \langle f, g + h \rangle = 0 + 8 = 8 \).

b) Compute \( \|g + h\| \).

**Solution:** \( \|g + h\|^2 = 1 + 2 \cdot 3 + 50 = 57 \) so \( \|g + h\| = \sqrt{57} \).

c) Find \( \text{proj}_E h \). [Express your solution as linear combinations of \( f \) and \( g \).]

**Solution:** Since \( f \) and \( g \) are orthogonal, they are an orthogonal basis for \( E \). Thus \( \text{proj}_E h = af + bg \) for some constants \( a \) and \( b \), that is,

\[ h = af + bg + w, \quad (2) \]

for some \( w \perp E \). To find \( a \) and \( b \), as usual we take the inner product of both sides with \( f \) and \( g \) and get

\[ a = \frac{\langle h, f \rangle}{\|f\|^2} = \frac{8}{4} = 2, \quad b = \frac{\langle h, g \rangle}{\|g\|^2} = \frac{3}{1} = 3. \]

Therefore,

\[ \text{proj}_E h = 2f + 3g \]

d) Find an orthonormal basis of the span of \( f \), \( g \), and \( h \). [Express your results as linear combinations of \( f \), \( g \), and \( h \).]

**Solution:** Since \( f \) and \( g \) are orthogonal and, from equation (2), \( w \) is orthogonal to both \( f \) and \( g \), we find that \( f \), \( g \), and \( w \) are an orthogonal bases. To get an orthonormal basis we need only normalize these. From (2),

\[ \|h\|^2 = \|2f\|^2 + \|3g\|^2 + \|w\|^2 \]

so \( \|w\|^2 = 50 - 4 \cdot 4 - 9 \cdot 1 = 25 \). Therefore an orthonormal basis is

\[ e_1 := \frac{1}{2}f, \quad e_2 := g, \quad e_3 := \frac{1}{5}w = \frac{1}{5}(h - 2f - 3g). \]

7. [Like Bretscher, Sec. 5.5 #26 & 28]. Use the inner product \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx \).

Define

\[ f(x) = \begin{cases} 
-1 & \text{if } -\pi < x \leq 0, \\
1 & \text{if } 0 < x \leq \pi,
\end{cases} \]

and extend \( f \) to all of \( \mathbb{R} \) as period is with period \( 2\pi \): \( f(x + 2\pi) = f(x) \). This is called a **square wave**.

a) Compute the first \( N \) terms in the Fourier Series

\[ f(x) = A_0 + \sum_{k=1}^{N} [A_k \cos kx + B_k \sin kx] + R_N(x), \quad (3) \]
where the remainder, \( R_N(x) \), is orthogonal to \( 1, \cos kx, \sin \ell x, \ k, \ell = 1, 2, \ldots, N \).

**Solution:** We use that with this inner product, the functions

\[
1, \cos kx, \quad \text{and} \quad \sin \ell x, \quad k, \ell = 1, 2, 3, \ldots
\]

are orthogonal with \( \|1\|^2 = 2\pi \), and \( \|\cos kx\|^2 = \|\sin \ell x\|^2 = \pi \).

Then, taking the inner product of equation (3) with the \( \cos jx \)'s and \( \sin \ell x \)'s we obtain

\[
A_0 = \langle f, 1 \rangle_{2\pi}, \quad A_k = \langle f, \cos kx \rangle_{\pi}, \quad B_k = \langle f, \sin kx \rangle_{\pi}.
\]

Since our function \( f(x) \) is odd, by Problem 3 we know that \( A_k = 0 \) for \( k = 0, 1, 2, 3, \ldots \) and

\[
B_k = -\int_{-\pi}^{0} \frac{\sin kx}{\pi} \, dx + \int_{0}^{\pi} \frac{\sin kx}{\pi} \, dx
\]

\[
= 2 \int_{0}^{\pi} \frac{\sin kx}{\pi} \, dx = -2 \left. \frac{\cos kx}{\pi} \right|_{0}^{\pi}
\]

\[
= \frac{2}{\pi} [1 - (-1)^k] = \begin{cases} \frac{4}{k\pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}
\]

Therefore, for \( N = 2K + 1 \),

\[
f(x) = \frac{4}{\pi} \left[ \frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2K+1)x}{2K+1} \right] + R_{2K+1}(x),
\]

where \( R_N \) is orthogonal to the preceding terms.

See [http://mathworld.wolfram.com/FourierSeriesSquareWave.html](http://mathworld.wolfram.com/FourierSeriesSquareWave.html) for an interesting graph of how this Fourier series converges to the square wave.

**b)** Apply the Pythagorean Theorem 5.5.6 (page 343) to your answer.

**Solution:** Because the terms in equation (3) are orthogonal, the Pythagorean theorem gives

\[
\|f\|^2 = 2\pi |A_0|^2 + \pi \sum_{k=1}^{N} [\|A_k\|^2 + \|B_k\|^2] + \|R_N\|^2.
\]

Applied to this example, where \( \|f\|^2 = 2\pi \) and assuming (without proof here that \( \|R_N\| \to 0 \)) it gives:

\[
2\pi = \pi \frac{16}{\pi^2} \left[ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right],
\]

that is,

\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8},
\]

which one would not likely guess.
8. Compute the determinant of the upper triangular matrix

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & 0 & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix}
\]

[Do the cases \( n = 2 \) and \( n = 3 \) first.]

**Solution:** Expanding by minors using the first column gives

\[
\det A = a_{11}(-1)^{1+1}\det \begin{pmatrix}
a_{22} & a_{23} & \cdots & a_{2n} \\
0 & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}
\]

Repeating this we find that \( \det A = a_{11}a_{22}\cdots a_{nn} \), so for an upper (or lower) triangular matrix, the determinant is the product of the diagonal elements.

9. The \( n \times n \) matrices \( A \) and \( B \) are *similar* if there is and invertible \( n \times n \) matrix \( S \) so that \( B = SAS^{-1} \). If \( A \) and \( B \) are similar, show that \( \det B = \det A \).

**Solution:** Since for any \( n \times n \) matrices \( \det(MN) = (\det M)(\det N) = \det(NM) \), then

\[
\det B = \det(SAS^{-1}) = \det(S)\det(AS^{-1}) = \det S \det S^{-1} \det(A) = \det A.
\]

[Last revised: March 9, 2014]