Directions This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 4 problems (15 points each, so total is 60 points). Maximum score is thus 110 points.

Closed book, no calculators or computers– but you may use one 3” × 5” card with notes on both sides. Clarity and neatness count.

Part A: Five short answer questions (10 points each, so 50 points).

A–1. Which of the following sets are linear spaces? [If not, why not?]

a) In $\mathbb{R}^3$, the span of $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

b) The points $\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 5$.

c) The set of points $(x, y) \in \mathbb{R}^2$ with $y = 2x + x^2$.

d) The set of once differentiable solutions $u(x)$ of $u' + 3x^2u = 0$. [You are not being asked to solve this equation.]

e) The set of polynomials $p(x)$ of degree at most 2 with $p'(1) = 0$. 

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A–2. Let \( S \) be the linear space of \( 2 \times 2 \) matrices \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( 2a + d = 0 \). Find a basis and compute the dimension of \( S \).

A–3. Let \( S \) and \( T \) be linear spaces and \( L : S \to T \) be a linear map. Say \( \vec{v}_1 \) and \( \vec{v}_2 \) are (distinct!) solutions of the equations \( L\vec{x} = \vec{y}_1 \) while \( \vec{w} \) is a solution of \( L\vec{x} = \vec{y}_2 \). Answer the following in terms of \( \vec{v}_1 \), \( \vec{v}_2 \), and \( \vec{w} \).

a) Find some solution of \( L\vec{x} = 2\vec{y}_1 - 7\vec{y}_2 \).

b) Find another solution (other than \( \vec{w} \)) of \( L\vec{x} = \vec{y}_2 \).
A–4. Say you have matrices $A$ and $B$.
   a) If $A : \mathbb{R}^7 \to \mathbb{R}^7$, what are the possible dimensions of the kernel of $A$? The image of $A$?

   b) If $B : \mathbb{R}^3 \to \mathbb{R}^5$, what are the possible dimensions of the kernel of $B$? The image of $B$?

A–5. Give an example of $2 \times 2$ matrices $A$ and $B$ with $AB = 0$ but $A \neq 0$ and $B \neq 0$. 
PART B  Four questions, 15 points each (so 60 points total).

B–1. Let \( C = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \). [NOTE: In this problem, there is no partial credit for incorrect computations.]

a) Find the inverse of \( C \).

b) Find the inverse of \( C^2 \).
B–2. Define the linear maps $A$, $B$, and $C$ from $\mathbb{R}^2 \to \mathbb{R}^2$ by the rules

- $A$ rotates vectors by $\pi/2$ radians counterclockwise.
- $B$ reflects vectors across the vertical axis.
- $C$ orthogonal projection onto the vertical axis, so $(x_1, x_2) \to (0, x_2)$

Let $M$ be the linear map that first applies $A$, then $B$, and finally $C$. Find a matrix that represents $M$ in the standard basis for $\mathbb{R}^2$. 
B–3. Let the linear map \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) be specified by the matrix
\[
A := \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{pmatrix}.
\]

a) Find a basis for the kernel of \( A \).

b) Find a basis for the image of \( A \).

c) With the above matrix \( A \), is it possible to find an invertible \( 3 \times 3 \) matrix \( B \) so that the matrix \( AB \) is invertible?
B–4. Say you are given the four data points \((-1, 0), (1, 2), (4, -2),\) and \((5, 3)\). Find a polynomial \(p(x)\) of degree at most three that passes through these four points. [Don’t bother to “simplify” your answer.]