Math 312  Exam 2  Jerry L. Kazdan
March 21, 2013  9:00 – 10:20

DIRECTIONS  This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 5 problems (15 points each, so total is 75 points). Maximum score is 125 points.

Closed book, no calculators or computers– but you may use one 3″ × 5″ card with notes on both sides. Clarity and neatness count.

PART A: Five short answer questions (10 points each, so 50 points).

A–1. Let \( A \) be a \( 5 \times 5 \) real matrix with \( \det A = -1 \). What is \( \det(-2A) \)?

A–2. We consider the equation \( Ax = b \) where \( x \) and \( b \) are in \( \mathbb{R}^4 \) and \( A \) is a \( 4 \times 4 \) matrix with determinant 7. True or False – and Why?
   a) For some vector \( b \) the equation \( Ax = b \) has exactly one solution.
   b) For some vector \( b \) the equation \( Ax = b \) has infinitely many solutions.
   c) For some vector \( b \) the equation \( Ax = b \) has no solution.
   d) For all vectors \( b \) the equation \( Ax = b \) has at least one solution.
A–3. A matrix is *nilpotent* if \( A^k = 0 \) for some positive integer \( k \) [Example: \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)]

If \( \lambda \) is an eigenvalue of a nilpotent matrix, show that \( \lambda = 0 \) [SUGGESTION: Begin with \( A\vec{v} = \lambda \vec{v} \)].

A–4. In \( \mathbb{R}^4 \), find the distance from the point \((1, -2, 0, 3)\) to the “plane” \( x_1 + 3x_2 - x_3 + x_4 = 0 \).

A–5. In \( \mathbb{R}^n \) with the usual inner product, show that

\[
\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 = 4\langle \vec{x}, \vec{y} \rangle
\]
Part B  Five questions, 15 points each (so 75 points total).

B–1. Find the eigenvalues and eigenvectors of the matrix \( A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \).
The matrix $B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Use this to solve the differential equation $\frac{d\vec{x}(t)}{dt} = B\vec{x}(t)$ with initial condition $\vec{x}(0) = (2, 0)$. 

For certain polynomials \( f \), \( g \), and \( h \) say we are given the following table of inner products:

<table>
<thead>
<tr>
<th>( \langle \cdot, \cdot \rangle )</th>
<th>( f )</th>
<th>( g )</th>
<th>( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>( g )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( h )</td>
<td>8</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>

For example, \( \langle g, h \rangle = \langle h, g \rangle = 0 \). Let \( E \) be the span of \( f \) and \( g \).

a) Compute \( \langle f, g + h \rangle \).

b) Compute \( \| g + h \| \).

c) Find \( \text{Proj}_E h \). [Express your solution as linear combinations of \( f \) and \( g \).]

d) Find an orthonormal basis of the span of \( f \), \( g \), and \( h \). [Express your results as linear combinations of \( f \), \( g \), and \( h \).]
B–4. Consider the space $\mathcal{P}_2$ of polynomials of degree at most two with the following inner product:

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

a) Compute the inner product of the polynomials $p(x) := 1$ and $q(x) := x$.

b) Using this inner product, find an orthogonal basis for the space $\mathcal{P}_2$. 

B–5. Say you have done an experiment and obtained the data points \((-1,1), (0,-1), (1,-1),\) and \((2,3)\). Based on some other evidence you believe this data should fit a curve of the form 
\[ y = a + bx^2. \]

a) Write the (over-determined) system of linear equations you would ideally like to solve for the unknown coefficients \(a\) and \(b\).

b) Use the method of least squares to find the normal equations for the coefficients \(a\) and \(b\).

c) Solve the normal equations to find the coefficients \(a\) and \(b\) explicitly (numbers, like \(3/5\) and \(-2\)).