Multiple Integral: Change of Variable

Say we have a multiple integral

\[ K := \int_{\mathbb{R}^2} \frac{1}{[1 + (x + 2y - 1)^2 + (3x + y + 2)^2]^2} \, dx \, dy \tag{1} \]

and would like to make the change of variable

\[ u = x + 2y - 1, \quad v = 3x + y + 2 \tag{2} \]

since that would clean-up the integrand. How is this done?

Here is the general rule for

\[ J := \int_{\mathcal{D}} h(v_1, v_2) \, dv_1 \, dv_2 \]

under the change of variable \( \bar{v} = F(\bar{u}) \) where \( F(\bar{u}) = \begin{pmatrix} f_1(\bar{u}) \\ f_2(\bar{u}) \end{pmatrix} \) is given by

\[ v_1 = f_1(u_1, u_2) \quad v_2 = f_2(u_1, u_2). \]

Note that here we have defined the old variables, \((v_1, v_2)\) in terms of the new variables, \((u_1, u_2)\), while in equations (1)-(2) we defined the new variables, \((u, v)\) in terms of the old ones, \((x, y)\). In practice, one uses whichever is more convenient.

To begin, compute the first derivative (or Jacobian) matrix:

\[ F'(\bar{u}) := \begin{pmatrix} \frac{\partial f_1(u_1, u_2)}{\partial u_1} & \frac{\partial f_1(u_1, u_2)}{\partial u_2} \\ \frac{\partial f_2(u_1, u_2)}{\partial u_1} & \frac{\partial f_2(u_1, u_2)}{\partial u_2} \end{pmatrix}. \tag{3} \]

Then the rule is

\[ dv_1 \, dv_2 = |\det F'(\bar{u})| \, du_1 \, du_2 \]

so in the new variables

\[ J = \int_{\mathcal{D}'} h(f_1(u_1, u_2), f_2(u_1, u_2)) \cdot |\det(F'(\bar{u}))| \, du_1 \, du_2, \]

where \( \mathcal{D}' \) is the region in the \( u_1u_2 \) plane corresponding to \( \mathcal{D} \).

**Example 1** Compute \( \int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy \).

We change to polar coordinates \( \begin{pmatrix} x \\ y \end{pmatrix} = F(r, \theta) \) with the usual formulas

\[ x = r \cos \theta \quad y = r \sin \theta. \]
Then, as in equation (3), the first derivative matrix is
\[
F'(r, \theta) = \begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}.
\]

Since \( \det F'(r, \theta) = r \) we have \( dx\, dy = r\, dr\, d\theta \) so
\[
\int_0^{2\pi} \int_0^\infty \frac{1}{(1 + r^2)^2} \, r \, dr \, d\theta = 2\pi \int_0^\infty \frac{1}{(1 + r^2)^2} \, r \, dr = \pi.
\]

**Example 2**

For the integral in equation (1)-(2) if we write \( \begin{pmatrix} u \\ v \end{pmatrix} = G(x, y) \) then the first derivative matrix is
\[
G'(x, y) = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix} = \begin{pmatrix} 1 & 2 \\
3 & 1 \end{pmatrix}
\]
so \( du\, dv = 5\, dx\, dy \).

Therefore, using polar coordinates, from equation (4)
\[
K = \iint_{\mathbb{R}^2} \frac{1}{(1 + (x + 2y - 1)^2 + (3x + y + 2)^2)^2} \, dx\, dy = \frac{\pi}{5}.
\]
Example 3 Compute \( J = \int_{R^2} \frac{1}{(1+2x_1^2+6x_1x_2+9x_2^2)^2} \, dx_1 \, dx_2. \)

SOLUTION Write \( 2x_1^2 + 6x_1x_2 + 9x_2^2 = \langle x, Ax \rangle, \) where \( A = \begin{pmatrix} 2 & 3 \\ 3 & 9 \end{pmatrix}. \) Idea: If \( A \) were the identity matrix, this would be straightforward, just use polar coordinates as in equation (4). Diagonalizing \( A \) is thus the essential step.

Since \( A \) is symmetric, it is orthogonally similar to a diagonal matrix, \( A = RDR^* \), where \( D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) has the eigenvalues of \( A \) on its diagonal and \( R \) is an orthogonal matrix.

\[
\langle x, Ax \rangle = \langle x, RDR^*x \rangle = \langle R^*x, DR^*x \rangle.
\]

Make the change of variable \( y = R^*x. \) In the integral, since \( |\det R| = 1, \) then, by (6),

\[
dy_1 \, dy_2 = |\det R^*| \, dx_1 \, dx_2 = dx_1 \, dx_2
\]

we find

\[
J = \int_{R^2} \frac{1}{(1+\lambda_1y_1^2+\lambda_2y_2^2)^2} \, dy_1 \, dy_2.
\]

Because \( A \) is positive definite (there is a simple test for \( 2 \times 2 \) matrices), its eigenvalues are positive so we make the further change of variable \( z_j = \sqrt{\lambda_j}y_j. \) This gives

\[
\lambda_1y_1^2 + \lambda_2y_2^2 = z_1^2 + z_2^2.
\]

and

\[
dz_1 \, dz_2 = \sqrt{\lambda_1\lambda_2} \, dy_1 \, dy_2 = \sqrt{|\det A|} \, dy_1 \, dy_2 = 3 \, dy_1 \, dy_2.
\]

Thus, as in equation (4),

\[
J = \frac{1}{3} \int_{R^2} \frac{1}{(1+z_1^2+z_2^2)^2} \, dz_1 \, dz_2 = \frac{\pi}{3}.
\]

It is interesting that although we used the theory that we could orthogonally diagonalize \( A, \) we never needed to compute explicitly its eigenvalues or eigenvectors.

ALTERNATE For this and other examples where \( \langle x, Ax \rangle \) with \( A \) positive definite arise, it is often faster (and clearer) to use that \( A \) has a positive definite square root, that is, there is a positive definite (symmetric) matrix \( B \) with \( A = B^2. \) Then

\[
\langle x, Ax \rangle = \langle x, B^2x \rangle = \langle Bx, Bx \rangle = \|Bx\|^2,
\]

which suggests making the change of variables \( y = Bx \) to find

\[
\langle x, Ax \rangle = \|y\|^2.
\]

If we use this approach in the above integral, then \( dy_1 \, dy_2 = |\det B| \, dx_1 \, dx_2 = \sqrt{|\det A|} \, dx_1 \, dx_2 \) so

\[
J = \frac{1}{\sqrt{|\det A|}} \int_{R^2} \frac{1}{(1+\|y\|^2)^2} \, dy_1 \, dy_2.
\]

As before, we now use polar coordinates (equation (4)) to conclude

\[
J = \frac{1}{3} \int_0^{2\pi} \left( \int_0^{\infty} \frac{1}{(1+r^2)^2} \, r \, dr \right) \, d\theta = \frac{\pi}{3}.
\]

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