Multiple Integral: Change of Variable

Say we have a multiple integral
\[ K := \int_{\mathbb{R}^2} \frac{1}{[1 + (x + 2y - 1)^2 + (3x + y + 2)^2]^2} \, dxdy \]  
and would like to make the change of variable
\[ u = x + 2y - 1, \quad v = 3x + y + 2 \]  
since that would clean-up the integrand. How is this done?

Here is the general rule for
\[ J := \int_{\mathcal{D}} h(v_1, v_2) \, dv_1 dv_2 \]  
under the change of variable \( \vec{v} = F(\vec{u}) \) where \( F(\vec{u}) = \begin{pmatrix} f_1(\vec{u}) \\ f_2(\vec{u}) \end{pmatrix} \) is given by
\[ v_1 = f_1(u_1, u_2) \quad v_2 = f_2(u_1, u_2). \]

Note that here we have defined the old variables, \((v_1, v_2)\) in terms of the new variables, \((u_1, u_2)\), while in equations (1)-(2) we defined the new variables, \((u, v)\) in terms of the old ones, \((x, y)\). In practice, one uses whichever is more convenient.

To begin, compute the first derivative (or Jacobian) matrix:
\[ F'(\vec{u}) := \begin{pmatrix} \frac{\partial f_1(u_1, u_2)}{\partial u_1} & \frac{\partial f_1(u_1, u_2)}{\partial u_2} \\ \frac{\partial f_2(u_1, u_2)}{\partial u_1} & \frac{\partial f_2(u_1, u_2)}{\partial u_2} \end{pmatrix}. \]

Then the rule is
\[ dv_1 \, dv_2 = |\det F'(\vec{u})| \, du_1 du_2 \]
so in the new variables
\[ J = \int_{\mathcal{D}'} h(f_1(u_1, u_2), f_2(u_1, u_2)) \, |\det(F'(\vec{u}))| \, du_1 du_2, \]
where \( D' \) is the region in the \( u_1u_2 \) plane corresponding to \( D \).

**Example 1** Compute \( \int \int_{R^2} \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy \).

We change to polar coordinates \( (x, y) = F(r, \theta) \) with the usual formulas

\[
x = r \cos \theta \quad y = r \sin \theta.
\]

Then, as in equation (3), the first derivative matrix is

\[
F'(r, \theta) = \begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}.
\]

Since \( \det F'(r, \theta) = r \) we have \( dx \, dy = r \, dr \, d\theta \) so

\[
\int \int_{R^2} \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy = \int_0^{2\pi} \left( \int_0^\infty \frac{1}{(1 + r^2)^2} r \, dr \right) \, d\theta
\]

\[= 2\pi \int_0^\infty \frac{1}{(1 + r^2)^2} r \, dr = \pi \tag{4}\]

**Example 2** For the integral in equation (1)-(2) if we write \( \begin{pmatrix} u \\ v \end{pmatrix} = G(x, y) \) then the first derivative matrix is

\[
G'(x, y) = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix} = \begin{pmatrix}
1 & 2 \\
3 & 1
\end{pmatrix}
\]

so \( du \, dv = 5 \, dx \, dy \).

Therefore, using polar coordinates, from equation (4)

\[
K = \int \int_{\mathbb{R}^2} \frac{1}{[1 + (x + 2y - 1)^2 + (3x + y + 2)^2]^2} \, dx \, dy
\]

\[= \int \int_{\mathbb{R}^2} \frac{1}{(1 + u^2 + v^2)^2} \, du \, dv = \frac{\pi}{5} \tag{5}\]

The identical procedure works in in higher dimensions. In \( \mathbb{R}^n \) say we have a multiple integral

\[
J := \int \cdots \int_D h(v_1, \ldots, v_n) \, dv_1 \cdots dv_n
\]
and want to make the change of variable \( \vec{v} = F(\vec{u}) \). As above, compute the first derivative matrix

\[
F'(\vec{u}) = \begin{pmatrix}
\frac{\partial f_1(\vec{u})}{\partial u_1} & \cdots & \frac{\partial f_1(\vec{u})}{\partial u_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n(\vec{u})}{\partial u_1} & \cdots & \frac{\partial f_n(\vec{u})}{\partial u_n}
\end{pmatrix}
\]

Then the element of “volume” becomes

\[
dv_1 \cdots dv_n = |\det F'(\vec{u})| du_1 \cdots du_n.
\]

This is particularly simple if one makes a linear change of variable, \( \vec{v} = A\vec{\bar{u}} \) where \( A \) is an invertible matrix whose elements are constants, so \( F(\vec{u}) = A\vec{\bar{u}} \). Then \( F'(\vec{u}) = A \) and we obtain

\[
dv_1 \cdots dv_n = |\det A| du_1 \cdots du_n \tag{6}
\]

and the change of variable formula is simply

\[
J := \int \cdots \int_D h(\vec{v}) dv_1 \cdots dv_n = \int \cdots \int_{\bar{D}} h(A\vec{\bar{u}}) |\det A| du_1 \cdots du_n.
\]

Example 3 Compute \( J = \iiint_{\mathbb{R}^2} \frac{1}{(1 + 2x_1^2 + 6x_1x_2 + 9x_2^2)^2} \, dx_1 \, dx_2 \).

Solution Write \( 2x_1^2 + 6x_1x_2 + 9x_2^2 = \langle \mathbf{x}, A\mathbf{x} \rangle \), where \( A = \begin{pmatrix} 2 & 3 \\ 3 & 9 \end{pmatrix} \). Idea: If \( A \) were the identity matrix, this would be straightforward, just use polar coordinates as in equation (4). Diagonalizing \( A \) is thus the essential step. Since \( A \) is symmetric, it is orthogonally similar to a diagonal matrix, \( A = RDR^* \), where \( D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) has the eigenvalues of \( A \) on its diagonal and \( R \) is an orthogonal matrix.

\[
\langle \mathbf{x}, A\mathbf{x} \rangle = \langle \mathbf{x}, RDR^*\mathbf{x} \rangle = \langle R^*\mathbf{x}, DR^*\mathbf{x} \rangle.
\]

Make the change of variable \( \mathbf{y} = R^*\mathbf{x} \). In the integral, since \( |\det R| = 1 \), then, by (6),

\[
dy_1 \, dy_2 = |\det R^*| dx_1 \, dx_2 = dx_1 \, dx_2
\]
we find

\[ J = \iint_{R^2} \frac{1}{(1 + \lambda_1 y_1^2 + \lambda_2 y_2^2)^2} \, dy_1 \, dy_2. \]

Because \( A \) is positive definite (there is a simple test for \( 2 \times 2 \) matrices), its eigenvalues are positive so we make the further change of variable \( z_j = \sqrt{\lambda_j} y_j \). This gives

\[ \lambda_1 y_1^2 + \lambda_2 y_2^2 = z_1^2 + z_2^2. \]

and

\[ dz_1 \, dz_2 = \sqrt{\lambda_1 \lambda_2} \, dy_1 \, dy_2 = \sqrt{\det A} \, dy_1 \, dy_2 = 3 dy_1 \, dy_2. \]

Thus, as in equation (4),

\[ J = \frac{1}{3} \iint_{R^2} \frac{1}{(1 + z_1^2 + z_2^2)^2} \, dz_1 \, dz_2 = \frac{\pi}{3}. \]

It is interesting that although we used the theory that we could orthogonally diagonalize \( A \), we never needed to compute explicitly its eigenvalues or eigenvectors.

**Alternate** For this and other examples where \( \langle x, Ax \rangle \) with \( A \) positive definite arise, it is often faster (and clearer) to use that \( A \) has a positive definite square root, that is, there is a positive definite (symmetric) matrix \( B \) with \( A = B^2 \). Then

\[ \langle x, Ax \rangle = \langle x, B^2 x \rangle = \langle Bx, Bx \rangle = \|Bx\|^2, \]

which suggests making the change of variables \( y = Bx \) to find

\[ \langle x, Ax \rangle = \|y\|^2. \]

If we use this approach in the above integral, then

\[ dy_1 \, dy_2 = |\det B| \, dx_1 \, dx_2 = \sqrt{|\det A|} \, dx_1 \, dx_2 \]

so

\[ J = \frac{1}{\sqrt{|\det A|}} \iint_{R^2} \frac{1}{(1 + \|y\|^2)^2} \, dy_1 \, dy_2. \]
As before, we now use polar coordinates (equation (4)) to conclude

\[ J = \frac{1}{3} \int_0^{2\pi} \left( \int_0^\infty \frac{1}{(1+r^2)} r \, dr \right) d\theta = \frac{\pi}{3}. \]

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