**ODE-Coupled**

As a mapping, the matrix $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an orthogonal reflection across the line $x_1 = x_2$. The eigenvectors $V$ have the property that $A\vec{v} = \lambda \vec{v}$ for some constant $\lambda$. On geometric grounds, under this reflection the points on this line $x_1 = x_2$ are fixed while the points on the line $x_2 = -x_1$ are reflected. In particular

$$A : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad A : \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix} = - \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

If we let $\vec{v}_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = \vec{v}_2$, so $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of $A$ with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. These vectors form a basis of $\mathbb{R}^2$ that is particularly useful to use for problems involving this matrix $A$.

To illustrate, we solve the differential equations

$$\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_1
\end{align*}$$

that is,

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad (1)$$

with initial conditions $x_1(0) = 4$ and $x_2(0) = 0$. In the above, $\vec{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. These equations are **coupled** since they both involve $x_1(t)$ and $x_2(t)$.

**METHOD 1** We use the eigenvectors of $A$ as our new basis

$$\vec{x}(t) = u_1(t)\vec{v}_1 + u_2(t)\vec{v}_2, \quad (2)$$
where the coefficients \( u_1(t) \) and \( u_2(t) \) are to be found. Substitute this into both sides of equation (1). Since neither \( \vec{v}_1 \) nor \( \vec{v}_2 \) depend on \( t \) we find:

\[
\frac{d\vec{x}(t)}{dt} = \frac{du_1(t)}{dt}\vec{v}_1 + \frac{du_2(t)}{dt}\vec{v}_2.
\]

Also, since the \( \vec{v}_j \) are eigenvectors of \( A \):

\[
A\vec{x} = u_1(t)A\vec{v}_1 + u_2(t)A\vec{v}_2 = u_1(t)\vec{v}_1 - u_2(t)\vec{v}_2.
\]

Thus, from equation (1)

\[
0 = \frac{d\vec{x}(t)}{dt} - A\vec{x}(t) = \left[ \frac{du_1(t)}{dt} - u_1(t) \right] \vec{v}_1 + \left[ \frac{du_2(t)}{dt} + u_2(t) \right] \vec{v}_2.
\]

Because \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent, their coefficients must both be zero:

\[
\frac{du_1(t)}{dt} = u_1(t) \quad \quad \quad \quad \frac{du_2(t)}{dt} = -u_2(t).
\]

Note these equations are **uncoupled** – and are easy to solve:

\[
u_1(t) = c_1 e^t \quad \quad \quad u_2(t) = c_2 e^{-t},
\]

where \( c_1 \) and \( c_2 \) are any constants. Shortly they will be determined by the initial conditions.

Substituting this into equation (2), we find that

\[
\vec{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{-t} \\ c_1 e^t - c_2 e^{-t} \end{pmatrix}.
\]

Now we use the initial condition to find the constants \( c_1 \) and \( c_2 \):

\[
\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix}.
\]

Therefore \( c_1 = c_2 = 2 \) so the desired solution is

\[
\vec{x}(t) = \begin{pmatrix} 2e^t + 2e^{-t} \\ 2e^t - 2e^{-t} \end{pmatrix},
\]
that is, 
\[ x_1(t) = 2e^t + 2e^{-t}, \quad x_2(t) = 2e^t - 2e^{-t}. \]

**METHOD 2** This is essentially identical, but here we explicitly introduce the change of coordinates \( S \) from the standard basis to the new basis consisting of the eigenvectors of \( A \). We want \( S^{-1}AS = D \) where \( D \) is the diagonal matrix consisting of the eigenvalues of \( A \), so

\[
D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

**Computational Note** If \( S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \), then

\[
SD = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 a & \lambda_2 b \\ \lambda_1 c & \lambda_2 d \end{pmatrix}
\]

so the columns of \( S \) are multiplied by the \( \lambda_j \)'s (\( DS \) multiplies the rows of \( S \) by the \( \lambda_j \)'s).

By general theory, the columns \( S \) are the corresponding eigenvectors of \( A \)

\[
S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Since \( A = SDS^{-1} \), we substitute this into equation (1)

\[
\frac{d\tilde{x}}{dt} = A\tilde{x} = SDS^{-1}\tilde{x}, \quad \text{that is,} \quad \frac{d(S^{-1}\tilde{x})}{dt} = DS^{-1}\tilde{x}
\]

and are let to make the change of variable \( \tilde{u} = S^{-1}\tilde{x} \) to find

\[
\frac{d\tilde{u}}{dt} = D\tilde{u}, \quad \text{that is,} \quad \frac{du_1}{dt} = u_1, \quad \frac{du_2}{dt} = -u_2.
\]

These are exactly the equations (3) we found above. Thus

\[
\tilde{u}(t) = \begin{pmatrix} c_1e^t \\ c_2e^{-t} \end{pmatrix},
\]
so, just as before,

$$\vec{x}(t) = S\vec{u}(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 e^{-t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{-t} \\ c_1 e^t - c_2 e^{-t} \end{pmatrix}. $$

Again, we can use the initial condition to find the constants $c_1$ and $c_2$.

**Exercise:** Say you have a sequence of vectors $\vec{x}_1, \vec{x}_2, \ldots$ with the property that $\vec{x}_{k+1} = A\vec{x}_k$, where $A$ is the above $2 \times 2$ matrix, and say the initial vector $X_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Compute $\vec{x}_k$ by using a basis consisting of the eigenvectors of $A$: $x_k = a_k \vec{v}_1 + b_k \vec{v}_2$.

Since our map $A$ is just an orthogonal reflection, without any computation (or mention of eigenvectors) the answer is obviously that if $k$ is even, then $\vec{x}_k = \vec{x}_0$ while if $k$ is odd, then $\vec{x}_k = \vec{x}_1$ is the reflected vector. The point of this problem is that the identical computation works in the general case where $A$ is any $n \times n$ matrix that can be diagonalized.