ODE-Matrix

Particular solution of $u'' - 3u = x^2$ using matrices. Let $\mathcal{P}_2$ be the linear space of polynomials of degree at most 2.

$$p(x) = a + bx + cx^2$$

Note that $L : \mathcal{P}_2 \to \mathcal{P}_2$. We want to represent $L$ as a $3 \times 3$ matrix using the standard basis for $\mathcal{P}_2$, which we indicate by $B$

$$p_0(x) = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B, \quad p_1(x) = x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_B, \quad p_2(x) = x^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_B.$$

Then

$$Lp_0 = L1 = -3 = -3p_0 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}_B,$$

$$Lp_1 = Lx = -3x = -3p_1 = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}_B,$$

$$Lp_2 = 2 - 3x^2 = 2p_0 - 3p_2 = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}_B.$$

In this basis, the matrix representing $L$ has $Lp_0$ as its first column, $Lp_1$ as its second column, and $Lp_2$ as its third column.

$$L_B = \begin{pmatrix} -3 & 0 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}_B.$$

To solve $u'' - 3u = x^2$ we solve the matrix equation

$$\begin{pmatrix} -3 & 0 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}_B \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1)$$

Of course this is exactly the same as seeking a solution of $u'' - 3u = x^2$ by seeking $u$ as $u(x) = a + bx + cx^2$ since $Lu = 2c - 3(a + bx + cx^2)$ so we want to pick the unknown coefficients $a, b, c$ so that

$$2c - 3(a + bx + cx^2) = x^2, \quad \text{that is,} \quad -3cx^2 - 3bx + (2c - 3a) = x^2$$

so

$$-3c = 1, \quad -3b = 0, \quad 2c - 3a = 0,$$

which is the same as (1)

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