Problem Set 9

Due: In class Thursday, Apr. 11. Late papers will be accepted until 1:00 PM Friday.

Lots of problems. Fortunately many are short.

1. This asks you to come up with four examples. In each case, find a real matrix (perhaps $2 \times 2$) that is:
   a) Both invertible and diagonalizable.
   b) Not invertible, but diagonalizable.
   c) Not diagonalizable but is invertible.
   d) Neither invertible nor diagonalizable.

2. Let $A := \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$.
   a) Find the eigenvalues of $A$.
   b) Is the origin a stable equilibrium of the discrete dynamical system $\vec{x}_{k+1} = A\vec{x}_k$? Explain.

3. [Bretscher, Sec. 7.5 #14] Let $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$. Find an invertible matrix $S$ so that $S^{-1}AS = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

4. [Bretscher, Sec. 7.6 #18] If $\vec{x}(t+1) = A\vec{x}(t)$, where $A := \begin{pmatrix} -0.8 & 0.6 \\ -0.8 & -0.8 \end{pmatrix}$ and $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find a real closed formula for the trajectory $\vec{x}(t)$. Also, draw a rough sketch.

5. [Bretscher, Sec. 7.5 #24] Find all the eigenvalues of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$.

6. [Bretscher, 5th ed Sec. 7.5 #32(a)] Consider the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$, where $A := \begin{pmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.4 \end{pmatrix}$, perhaps modeling the way people search a mini-web. Using technology (say the Maple example I did in class: [http://hans.math.upenn.edu/~kazdan/312S13/Maple/MarkovChain.mw](http://hans.math.upenn.edu/~kazdan/312S13/Maple/MarkovChain.mw)), compute high powers of $A$, say $A^6$, $A^{16}$ and $A^{32}$, and make a conjecture about $\lim_{t \to \infty} A^t$. 

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7. [Bretscher, Sec. 7.3 #28] Let \( B := \begin{pmatrix} k & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & k \end{pmatrix} \) where \( k \) is an arbitrary constant. Find the eigenvalue(s) of \( B \) and determine both their algebraic and geometric multiplicities. [*Note: First try the analogous \( 2 \times 2 \) case.*] 

8. Let \( A \) be an \( n \times n \) real matrix. If \( A \) is orthogonally similar to a real diagonal matrix \( D \), must \( A \) be symmetric? Proof or counterexample [The matrices \( A \) and \( B \) are *orthogonally similar* if \( A = RBR^{-1} \) for some orthogonal matrix \( R \).]

9. [Bretscher, Sec. 8.1 #24] Find an orthonormal eigenbasis for 
\[
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]

10. [Bretscher, Sec. 8.1 #38] Let \( A \) be a symmetric \( 2 \times 2 \) matrix with eigenvalues \(-2\) and \( 3 \) and \( u \in \mathbb{R}^2 \) any unit vector. What are the possible values of \( \langle u, Au \rangle \)? Illustrate your answer in terms of the unit circle and its image under \( A \).

11. Of the following three matrices, one can be orthogonally diagonalized; one can be diagonalized (but not orthogonally); and one cannot be diagonalized at all. Identify these – *fully explaining your reasoning.*

\[
A = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.
\]

12. [Bretscher, Sec. 8.2 #18] Sketch the curve of points in the plane that satisfy 
\[9x_1^2 - 4x_1x_2 + 6x_2^2 = 1.\]

13. a) Let \( D := \begin{pmatrix} 4 & 0 \\ 0 & 25 \end{pmatrix} \) Find a positive definite symmetric matrix \( P \) so that \( P^2 = D \)
    (we call \( P \) the *square root* of \( D \))

b) Let \( A := \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} \). Find a positive definite (symmetric) matrix \( P \) so that \( P^2 = A \).

c) Show that every positive definite symmetric matrix \( A \) has a positive definite square root.
14. [Bretscher, Sec. 8.2 #28] Show that any positive definite $n \times n$ matrix $A$ can be written as $A = BB^*$, where the columns of $B$ are orthogonal. [HINT: Use the result of the previous problem.]

15. [Bretscher, Sec. 8.2 #26] Consider the quadratic polynomial $Q(x) := \langle x, Ax \rangle$, where $A$ is a real $n \times n$ symmetric matrix. “If for some vector $\bar{v} \neq 0$ we know that $Q(\bar{v}) = 0$, then $A$ cannot be invertible.” Proof or counterexample.

16. Let $f(x, y) := (x^2 + 4y^2)e^{(1-x^2-y^2)}$. Find and classify all of its critical points as local maxima etc.

[Last revised: May 5, 2013]