Directions: This exam has two parts. Part A has 4 shorter questions, (5 points each so total 20 points) while Part B had 6 problems (12 points each, so total is 72 points). Maximum score is thus 92 points.

Closed book, no calculators or computers– but you may use one 3” × 5” card with notes on both sides. Clarity and neatness count.

Part A: Four short answer questions (5 points each, so 20 points).

A–1. Let $A$ be a $3 \times 3$ real matrix two of whose eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1 - 2i$, with corresponding eigenvectors $v_1$ and $v_2$, what are $\lambda_3$ and $v_3$?

A–2. Given a unit vector $w \in \mathbb{R}^n$, let $W = \text{span}\{w\}$ and consider the linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$T(x) = 2 \text{Proj}_W(x) - x,$$

where $\text{Proj}_W(x)$ is the orthogonal projection onto $W$. Show that $T$ is one-to-one.
A–3. Let \( A \) be an invertible matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \) and corresponding eigenvectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \). What can you say about the eigenvalues and eigenvectors of \( A^{-1} \)? Justify your response.

A–4. Let \( A \) be an \( n \times n \) real self-adjoint matrix and \( \mathbf{v} \) an eigenvector with eigenvalue \( \lambda \). Let \( W = \text{span} \{ \mathbf{v} \} \).

a) If \( \mathbf{w} \in W \), show that \( A\mathbf{w} \in W \).

b) If \( \mathbf{z} \in W^\perp \), show that \( A\mathbf{z} \in W^\perp \).

Part B  Six questions, 12 points each (so 72 points total).

B–1. Let \( A \) be a real symmetric matrix. Say that \( \vec{v}_1 \) and \( \vec{v}_2 \) are eigenvectors corresponding to distinct eigenvalues \( \lambda_1 \neq \lambda_2 \). Show that \( \vec{v}_1 \) and \( \vec{v}_2 \) are orthogonal.
B–2. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs.

One has data on which location the cars are returned daily:

- **Rented at Airport**: 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
- **Rented in City**: 10% are returned to Airport, 10% returned to Suburbs.
- **Rented in Suburbs**: 20% are returned to the Airport and 5% to the City.

If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?
B–3. Let \( A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \).

a) What is the dimension of the image of \( A \)? Why?

b) What is the dimension of the kernel of \( A \)? Why?

c) What are the eigenvalues of \( A \)? Why?

d) What are the eigenvalues of \( B := \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 1 & 1 & 5 \end{pmatrix} \)? Why? [HINT: \( B = A + 3I \)].
B–4. For certain polynomials \( p(t), q(t), \) and \( r(t), \) say we are given the following table of inner products:

\[
\begin{array}{|c|ccc|}
\hline
\langle \, , \rangle & p & q & r \\
\hline
p & 4 & 0 & 8 \\
q & 0 & 1 & 0 \\
r & 8 & 0 & 50 \\
\hline
\end{array}
\]

For example, \( \langle q, r \rangle = \langle r, q \rangle = 0. \) Let \( E \) be the span of \( p \) and \( q. \)

a) Compute \( \langle p, q + r \rangle. \)

b) Compute \( \|q + r\|. \)

c) Find the orthogonal projection \( \text{Proj}_E r. \) [Express your solution as linear combinations of \( p \) and \( q. \)]

d) Find an orthonormal basis of the span of \( p, q, \) and \( r. \) [Express your results as linear combinations of \( p, q, \) and \( r. \)]
B–5. An $n \times n$ matrix is called *nilpotent* if $A^k$ equals the zero matrix for some positive integer $k$. (For instance, \(egin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) is nilpotent.)

a) If $\lambda$ is an eigenvalue of a nilpotent matrix $A$, show that $\lambda = 0$. (Hint: start with the equation $A\vec{x} = \lambda \vec{x}$.)

b) Show that if $A$ is both nilpotent and diagonalizable, then $A$ is the zero matrix. [Hint: use Part a].

c) Let $A$ be the matrix that represents $T : \mathcal{P}_5 \rightarrow \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $Tp = dp/dx$. Without doing any computations, explain why $A$ must be nilpotent.
B–6. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Show that
\[
\dim(\ker A) - \dim(\ker A^*) = k - n.
\]
In particular, for a square matrix, $\dim(\ker A) = \dim(\ker A^*)$. 