Directions  This exam has two parts. Part A has 4 shorter questions, (5 points each so total 20 points) while Part B had 6 problems (12 points each, so total is 72 points). Maximum score is thus 92 points.
Closed book, no calculators or computers– but you may use one 3′′ × 5′′ card with notes on both sides. Clarity and neatness count.

Part A: Four short answer questions (5 points each, so 20 points).

A–1. Let \( A \) be a 3 × 3 real matrix two of whose eigenvalues are \( \lambda_1 = -2 \) and \( \lambda_2 = 1 - 2i \), with corresponding eigenvectors \( v_1 \) and \( v_2 \), what are \( \lambda_3 \) and \( v_3 \)?

Solution  We know that complex eigenvalues come in pairs i.e. \( \lambda_3 = \overline{\lambda_2} = 1 + 2i \) and \( A v_2 = \overline{A} v_2 = \overline{\lambda_2} v_2 \) hence \( v_3 = v_2 \).

A–2. Given a unit vector \( w \in \mathbb{R}^n \), let \( W = \text{span} \{ w \} \) and consider the linear map \( T : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
T(x) = 2 \text{Proj}_W(x) - x,
\]
where \( \text{Proj}_W(x) \) is the orthogonal projection onto \( W \). Show that \( T \) is one-to-one.

Method 1  We need to show that the kernel of \( T \) is trivial, so we need to solve:

\[
2 \text{Proj}_W(x) - x = 0 \tag{1}
\]

To the above equation we apply \( T \) again and obtain:

\[
0 = T(2 \text{Proj}_W(x) - x) = 2 \text{Proj}_W(2 \text{Proj}_W(x) - x) - 2 \text{Proj}_W(x) + x
\]

so:

\[
0 = 4 \text{Proj}_W(x) - 2 \text{Proj}_W(x) - 2 \text{Proj}_W(x) + x = x
\]

Hence, the kernel of \( T \) is trivial, namely \( T \) is one-to-one.

Method 2  Since \( w \) is a unit vector, \( \text{Proj}_W(x) = \langle x, w \rangle w \) so equation (1) is

\[
2\langle x, w \rangle w = x.
\]

Taking the inner product of this with \( w \) gives \( 2\langle x, w \rangle = \langle x, w \rangle \) so \( \langle x, w \rangle = 0 \). Equation (1) then gives \( x = 0 \).

Method 3  Let \( P : \mathbb{R}^n \to \mathbb{R}^n \) be any projection, not necessarily orthogonal. It has the property \( P^2 = P \). Define

\[
T x := c P x - x
\]

for any constant \( c \). Claim: if \( c \neq 1 \), then \( \ker T = 0 \) (so \( T \) is one-to-one). To see this, apply \( P \) to both sides of \( c P x = x \) and use \( P^2 = P \) to find \( c P x = P x \). Because \( c \neq 1 \), then \( P x = 0 \). Consequently \( x = 0 \).
A–3. Let $A$ be an invertible matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. What can you say about the eigenvalues and eigenvectors of $A^{-1}$? Justify your response.

Solution Since $A$ invertible we have that $A\vec{v}_i = \lambda_i \vec{v}_i$ and $\lambda_i \neq 0$ for all $i$. Hence by multiplying $\frac{1}{\lambda_i}A^{-1}$ on both sides of $A\vec{v}_i = \lambda_i \vec{v}_i$ we obtain that $A^{-1}\vec{v}_i = \frac{1}{\lambda_i} \vec{v}_i$. So $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}$ are the eigenvalues of $A^{-1}$ with the same corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

A–4. Let $A$ be an $n \times n$ real self-adjoint matrix and $\vec{v}$ an eigenvector with eigenvalue $\lambda$. Let $W = \text{span} \{\vec{v}\}$.

a) If $w \in W$, show that $Aw \in W$.

Solution If $w \in W$ then $w = k\vec{v}$. Hence $Aw = Ak\vec{v} = k\lambda \vec{v} \in W$.

b) If $z \in W^\perp$, show that $Az \in W^\perp$.

Solution If $z \in W^\perp$ then $\langle z, \vec{v} \rangle = 0$. Hence $\langle Az, \vec{v} \rangle = \langle z, A^*\vec{v} \rangle = \langle z, A\vec{v} \rangle = \langle z, \lambda \vec{v} \rangle = \lambda \langle z, \vec{v} \rangle = 0$ so $Az \in W^\perp$.

PART B Six questions, 12 points each (so 72 points total).

B–1. Let $A$ be a real symmetric matrix. Say that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$. Show that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

Solution We have that:

$$\lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle = \langle A\vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, A^*\vec{v}_2 \rangle = \langle \vec{v}_1, A\vec{v}_2 \rangle = \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle$$

$$(\lambda_1 - \lambda_2)\langle \vec{v}_1, \vec{v}_2 \rangle = 0$$

so $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$, namely $\vec{v}_1$, $\vec{v}_2$ are orthogonal.

Method 2 Since $\lambda_1 \neq \lambda_2$, at least one of them is not zero. Say $\lambda_2 \neq 0$. Now use

$$\langle A\vec{v}_1, A\vec{v}_2 \rangle = \lambda_1 \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle$$

and

$$\langle A^2\vec{v}_1, A\vec{v}_2 \rangle = \lambda_2^2 \langle \vec{v}_1, \vec{v}_2 \rangle = \lambda_2 \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle = \lambda_2 \langle A\vec{v}_1, A\vec{v}_2 \rangle = \lambda_2 \lambda_1 \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle$$

Now use $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$ to conclude $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$.

B–2. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- **Rented at Airport:** 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
- **Rented in City:** 10% are returned to Airport, 10% returned to Suburbs.
- **Rented in Suburbs:** 20% are returned to the Airport and 5% to the City.
If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?

**SOLUTION** The equations we obtain from the information given is:

\[ x_{k+1} = 0.75x_k + 0.1y_k + 0.2z_k \]
\[ y_{k+1} = 0.05x_k + 0.8y_k + 0.05z_k \]
\[ z_{k+1} = 0.2x_k + 0.1y_k + 0.75z_k \]

where \( x \)'s, \( y \)'s, \( z \)'s correspond to information about cars rented at airport, city, suburbs respectively. Hence the transition matrix is:

\[
T = \begin{pmatrix}
0.75 & 0 & 0.2 \\
0.05 & 0.8 & 0.05 \\
0.2 & 0.1 & 0.75
\end{pmatrix}
\]

which is regular, so we need to find the probability eigenvector corresponding to the eigenvalue \( \lambda = 1 \). Solving \( T\vec{v} = \vec{v} \) we obtain \( v_1 = v_3 \) and \( v_2 = 0.5v_3 \) where \( \vec{v} = (v_1, v_2, v_3) \). Hence a eigenvector corresponding to \( \lambda = 1 \) is:

\[
\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}
\]

so the unique probability eigenvector corresponding to \( \lambda = 1 \) is:

\[
1/5\vec{v} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix}.
\]

Now, initially there were 100 cars so the long term distribution is: 40 cars at the Airport, 20 at the City and 40 at the Suburbs.

**B–3.** Let \( A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \).

a) What is the dimension of the image of \( A \)? Why?

**SOLUTION** Since \( \text{im} \ A \) is the column-space of \( A \) we have that \( \text{im} \ A = \text{span} \{(1,1,1)\} \), so \( \text{dim} \ (\text{im} \ A) = 1 \).

b) What is the dimension of the kernel of \( A \)? Why?

**SOLUTION** From rank-nullity theorem and part (a) we have that \( \text{dim} \ (\ker \ A) = 2 \).

c) What are the eigenvalues of \( A \)? Why?

**SOLUTION 1:** Since \( \ker \ A \) is 2-dimensional it implies that two of the eigenvalues of \( A \) are 0. Also since the trace of \( A \) (which is equal to 4) is equal to the sum of its eigenvalues we have that the third eigenvalue is equal to 4.

**SOLUTION 2:** Using the characteristic polynomial of \( A \) which is: \( p_A(\lambda) = \lambda^2(4 - \lambda) \).
d) What are the eigenvalues of $B := \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 1 & 1 & 5 \end{pmatrix}$? Why? [HINT: $B = A + 3I$].

**Solution** If $\lambda$ is an eigenvalue of $A$ and $v$ the corresponding eigenvector then:

$$Bv = (A + 3I)v = (\lambda + 3)v$$

hence using part (c) we obtain that the eigenvalues of $B$ are 3, 3, 7.

B–4. For certain polynomials $p(t)$, $q(t)$, and $r(t)$, say we are given the following table of inner products:

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$q$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$r$</td>
<td>8</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>

For example, $\langle q, r \rangle = \langle r, q \rangle = 0$. Let $E$ be the span of $p$ and $q$.

a) Compute $\langle p, q + r \rangle$.

**Solution** $\langle p, q + r \rangle = \langle p, q \rangle + \langle p, r \rangle = 0 + 8 = 8$

b) Compute $\|q + r\|$.

**Solution** $\|q + r\| = \sqrt{\langle q, q \rangle + \langle r, r \rangle + 2\langle q, r \rangle} = \sqrt{1 + 50 + 0} = \sqrt{51}$

c) Find the orthogonal projection $\text{Proj}_E r$. [Express your solution as linear combinations of $p$ and $q$.]

**Solution** $\text{Proj}_E r = \frac{\langle r, p \rangle}{\langle p, p \rangle} p + \frac{\langle r, q \rangle}{\langle q, q \rangle} q = 2p$.

d) Find an orthonormal basis of the span of $p$, $q$, and $r$. [Express your results as linear combinations of $p$, $q$, and $r$.]

**Solution** We apply the Gram-Schmidt process to first get an orthogonal basis $\{u_1, Bu_2, Bu_3\}$ and then the orthonormal basis $\{e_1, e_2, e_3\}$:

$$u_1 = q \quad \text{and} \quad e_1 = q$$

$$u_2 = p - \frac{\langle p, q \rangle}{\langle q, q \rangle} q = p - \frac{1}{4}q \quad \text{and} \quad e_2 = \frac{1}{\sqrt{17}}p$$

$$u_3 = r - \frac{\langle r, q \rangle}{\langle q, q \rangle} q - \frac{\langle r, p \rangle}{\langle p, p \rangle} p = r - \frac{2}{\sqrt{17}}p \quad \text{and} \quad e_3 = \frac{r - 2p}{\sqrt{34}}$$

since $\|r - 2p\|^2 = \langle r, r \rangle + 4\langle p, p \rangle - 4\langle r, p \rangle = 50 + 16 - 32 = 34$.

B–5. An $n \times n$ matrix is called nilpotent if $A^k$ equals the zero matrix for some positive integer $k$. (For instance, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent.)
a) If $\lambda$ is an eigenvalue of a nilpotent matrix $A$, show that $\lambda = 0$. (Hint: start with the equation $A\vec{x} = \lambda \vec{x}$.)

**Solution** We have $A\vec{x} = \lambda \vec{x}$ so $A^k\vec{x} = \lambda^k \vec{x}$. Hence $\lambda^k \vec{x} = 0$ so $\lambda = 0$ since $\vec{x} \neq 0$ (because it is an eigenvector).

b) Show that if $A$ is both nilpotent and diagonalizable, then $A$ is the zero matrix. [Hint: use Part a].

**Solution** From part (a) we deduce that all eigenvalues of $A$ are zero, hence $A$ is similar to the zero matrix hence $A = S(0)S^{-1} = 0$ where $0$ the zero matrix and $S$ some matrix.

c) Let $A$ be the matrix that represents $T : \mathcal{P}_5 \to \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $Tp = dp/dx$. Without doing any computations, explain why $A$ must be nilpotent.

**Solution** Since $p$ polynomial of degree at most 5 we have that $T^6$ is the zero map ($T^6 = T \circ T \circ T \circ T \circ T \circ T$ composition of $T$ with itself) hence $A^6 = 0$ namely $A$ nilpotent.

---

B–6. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Show that 

$$\dim(\ker A) - \dim(\ker A^*) = k - n.$$ 

In particular, for a square matrix, $\dim(\ker A) = \dim(\ker A^*)$.

**Solution 1:** Since in $\mathbb{R}^k$, $(\text{im } A^*)^\perp = \ker A$, we have that 

$$\dim(\ker A) + \dim(\text{im } A^*) = k$$ 

Also, since $A^* : \mathbb{R}^n \to \mathbb{R}^k$, by the rank-nullity theorem 

$$\dim(\ker A^*) + \dim(\text{im } A^*) = n$$ 

Then we subtract to obtain:

$$\dim(\ker A) - \dim(A^*) = k - n.$$ 

**Solution 2:** Since $A^* : \mathbb{R}^n \to \mathbb{R}^k$, by a homework problem $\dim \text{im } A = \dim \text{im } A^*$. Using rank-nullity theorem we have:

$$\dim(\ker A) - \dim(\ker A^*) = \dim \mathbb{R}^k - \dim \text{im } A - (\dim \mathbb{R}^n - \dim \text{im } A^*) = k - n$$