Directions This exam has three parts. Part A has 5 shorter questions, (6 points each), Part B has 6 True/False questions (5 points each), and Part C has 5 standard problems (12 points each). Maximum score is thus 120 points.

Closed book, no calculators, cell phones, or computers– but you may use one 3" × 5" card with notes on both sides. Clarity and neatness count.

Part A: Five short answer questions (6 points each, so 30 points).

A–1. Suppose $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map represented by a matrix, $A$.
    
    a) What are the possible values for the rank of $A$? Why?
    b) What are the possible values for the dimension of the kernel of $A$? Why?
    c) Suppose the rank of $A$ is as large as possible. What is the dimension of $\ker(A)^\perp$? Explain.

A–2. In the following equations

$$
    \begin{align*}
    x_1 + x_2 + 2x_3 + x_4 &= 1 \\
    x_1 - x_2 - 2x_3 + x_4 &= 0 \\
    -x_1 + x_2 - 2x_3 + x_4 &= 3 \\
    -x_1 - x_2 + 2x_3 + x_4 &= 2
    \end{align*}
$$

solve for $x_2$ (only!). [Observe that if you write this as $x_1 v_1 + \cdots + x_4 v_4 = b$, then the vectors $v_j$ are orthogonal.]

A–3. Let $P_1 = (a_1, b_1), P_2 = (a_2, b_2), \ldots P_5 = (a_5, b_5)$ be five points in the plane $\mathbb{R}^2$. Find the point $Q = (x, y)$ that minimizes

$$
    f(x, y) = \|P_1 - Q\|^2 + \|P_2 - Q\|^2 + \cdots + \|P_5 - Q\|^2.
$$

A–4. Let $A$ be an $n \times k$ matrix.

    a) If $\lambda_1 \neq 0$ is an eigenvalue of $A^* A$, show that it is also an eigenvalue of $AA^*$. [Note where you use $\lambda_1 \neq 0$].
    b) If $\vec{v}_1$ and $\vec{v}_2$ are orthogonal eigenvectors of $A^* A$, let $\vec{u}_1 = A \vec{v}_1$, and $\vec{u}_2 = A \vec{v}_2$. Show that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.

A–5. Let $A$ be a real matrix with the property that $\langle \vec{x}, A \vec{x} \rangle = 0$ for all real vectors $\vec{x}$.

    a) If $A$ is a symmetric matrix, show this implies that $A = 0$.
    b) Give an example of a matrix $A \neq 0$ that satisfies $\langle \vec{x}, A \vec{x} \rangle = 0$ for all real vectors $\vec{x}$.
PART B  Six True or False questions (5 points each, so 30 points). Be sure to give a brief explanation.

B–1. If \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} is a collection of vectors in \(\mathbb{R}^5\), then the span of \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} must be a three-dimensional subspace of \(\mathbb{R}^5\).

B–2. The set of polynomials in \(\mathcal{P}_4\) satisfying \(p(0) = 2\) is a linear subspace of \(\mathcal{P}_4\).

B–3. If \(A: \mathbb{R}^k \to \mathbb{R}^n\) be a linear map and \(\ker A^* = 0\), then for any \(\vec{b} \in \mathbb{R}^n\) there is at least one solution of \(A\vec{x} = \vec{b}\).

B–4. If \(A\) is a \(3 \times 3\) matrix with eigenvalues 1, 2, and 4, then \(A - 4I\) is invertible.

B–5. If \(A\) is diagonalizable square matrix, then so is \(A^2\).

B–6. If a real matrix \(A\) can be orthogonally diagonalized, then it is self-adjoint (that is, symmetric).

PART C  Five questions, 12 points each (so 60 points total).

[Check your computation of any eigenvalues by computing the trace and determinant of the matrix].

C–1. Let \(A: \mathbb{R}^k \to \mathbb{R}^n\) be a linear map.
   a) If \(k = n\), so \(A\) is represented by a square matrix, show that \(\ker A = 0\) implies that \(A\) is also onto – and hence invertible.
   b) If \(k \neq n\), show that \(A\) cannot be invertible. NOTE there are two cases: \(k < n\) and \(k > n\).

C–2. a) Find an orthogonal matrix \(R\) that diagonalizes \(A := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\).

   b) Compute \(A^{50}\).

C–3. Of the following four matrices, which can be orthogonally diagonalized; which can be diagonalized (but not orthogonally); and which cannot be diagonalized at all. Identify these – fully explaining your reasoning.

\[ A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}. \]
C–4. Let \( A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & -1 \end{pmatrix} \). Find a vector \( \vec{v} \) that maximize \( \|A\vec{x}\| \) on the unit disk \( \|\vec{x}\| = 1 \). What is this maximum value?

C–5. Let \( \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \) be a solution of the system of differential equations

\[
\begin{align*}
x_1' &= cx_1 + x_2 \\
x_2' &= -x_1 + cx_2.
\end{align*}
\]

For which value(s) of the real constant \( c \) do all solutions \( \vec{x}(t) \) converge to 0 as \( t \to \infty \)?