Directions: This exam has three parts. Part A has 5 shorter questions, (6 points each), Part B has 6 True/False questions (5 points each), and Part C has 5 standard problems (12 points each). Maximum score is thus 120 points.

Closed book, no calculators or computers— but you may use one 3″ × 5″ card with notes on both sides. Clarity and neatness count.

Part A: Five short answer questions (6 points each, so 30 points).

A–1. Suppose \( T: \mathbb{R}^6 \to \mathbb{R}^4 \) is a linear map represented by a matrix, \( A \).

a) What are the possible values for the rank of \( A \)? Why?

b) What are the possible values for the dimension of the kernel of \( A \)? Why?

c) Suppose the rank of \( A \) is as large as possible. What is the dimension of \( \ker(A)^\perp \)? Explain.

A–2. In the following equations

\[
\begin{align*}
  x_1 + x_2 + 2x_3 + x_4 &= 1 \\
  x_1 - x_2 - 2x_3 + x_4 &= 0 \\
  -x_1 + x_2 - 2x_3 + x_4 &= 3 \\
  -x_1 - x_2 + 2x_3 + x_4 &= 2
\end{align*}
\]

solve for \( x_2 \) (only!). [Observe that if you write this as \( x_1 \vec{v}_1 + \cdots + x_4 \vec{v}_4 = \vec{b} \), then the vectors \( \vec{v}_j \) are orthogonal.]
A–3. Let \( P_1 = (a_1, b_1), \ P_2 = (a_2, b_2), \ldots P_5 = (a_5, b_5) \) be five points in the plane \( \mathbb{R}^2 \). Find the point \( Q = (x, y) \) that minimizes
\[
f(x, y) = \|P_1 - Q\|^2 + \|P_2 - Q\|^2 + \cdots + \|P_5 - Q\|^2.
\]

A–4. Let \( A \) be an \( n \times k \) matrix.

a) If \( \lambda_1 \neq 0 \) is an eigenvalue of \( A^* A \), show that it is also an eigenvalue of \( AA^* \). [Note where you use \( \lambda_1 \neq 0 \)].

b) If \( \vec{v}_1 \) and \( \vec{v}_2 \) are orthogonal eigenvectors of \( A^* A \), let \( \vec{u}_1 = A\vec{v}_1 \), and \( \vec{u}_2 = A\vec{v}_2 \). Show that \( \vec{u}_1 \) and \( \vec{u}_2 \) are orthogonal.
A–5. Let $A$ be a real matrix with the property that $\langle \vec{x}, A\vec{x} \rangle = 0$ for all real vectors $\vec{x}$.

a) If $A$ is a symmetric matrix, show this implies that $A = 0$.

b) Give an example of a matrix $A \neq 0$ that satisfies $\langle \vec{x}, A\vec{x} \rangle = 0$ for all real vectors $\vec{x}$. 
Part B  Six True or False questions (5 points each, so 30 points). Be sure to give a brief explanation.

B–1. If \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} is a collection of vectors in \(\mathbb{R}^5\), then the span of \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} must be a three-dimensional subspace of \(\mathbb{R}^5\).

B–2. The set of polynomials in \(\mathcal{P}_4\) satisfying \(p(0) = 2\) is a linear subspace of \(\mathcal{P}_4\).

B–3. If \(A : \mathbb{R}^k \rightarrow \mathbb{R}^n\) be a linear map and \(\ker A^* = 0\), then for any \(\vec{b} \in \mathbb{R}^n\) there is at least one solution of \(A\vec{x} = \vec{b}\).

B–4. If \(A\) is a \(3 \times 3\) matrix with eigenvalues 1, 2, and 4, then \(A - 4I\) is invertible.

B–5. If \(A\) is diagonalizable square matrix, then so is \(A^2\).

B–6. If a real matrix \(A\) can be orthogonally diagonalized, then it is self-adjoint (that is, symmetric).
PART C  Five questions, 12 points each (so 60 points total).

[Check your computation of any eigenvalues by computing the trace and determinant of the matrix].

C–1. Let \( A : \mathbb{R}^k \to \mathbb{R}^n \) be a linear map.

a) If \( k = n \), so \( A \) is represented by a square matrix, show that \( \ker A = 0 \) implies that \( A \) is also onto – and hence invertible.

b) If \( k \neq n \), show that \( A \) cannot be invertible. NOTE there are two cases: \( k < n \) and \( k > n \).
C–2. a) Find an orthogonal matrix $R$ that diagonalizes $A := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

b) Compute $A^{50}$. 
C–3. Of the following four matrices, which can be orthogonally diagonalized; which can be diagonalized (but not orthogonally); and which cannot be diagonalized at all. Identify these – fully explaining your reasoning.

\[ A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}. \]
C–4. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & -1 \end{pmatrix}$. Find a vector $\vec{v}$ that maximize $\|A\vec{x}\|$ on the unit disk $\|\vec{x}\| = 1$. What is this maximum value?
C-5. Let \( \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \) be a solution of the system of differential equations

\[
\begin{align*}
x_1' &= cx_1 + x_2 \\
x_2' &= -x_1 + cx_2.
\end{align*}
\]

For which value(s) of the real constant \( c \) do all solutions \( \vec{x}(t) \) converge to 0 as \( t \to \infty \)?