ODE-Diagonalize: Examples

**Example 1** Let \( A := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \) and \( \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \). Solve \( \frac{d\vec{x}}{dt} = A\vec{x} \) with \( \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). (1)

**Solution:** The key observation is that if \( A \) were a diagonal matrix, this would be simple. Thus we begin by finding the eigenvalues and eigenvectors of \( A \). By an easy calculation

\[
\det(A - \lambda I) = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5).
\]

Thus the eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = 5 \) with corresponding eigenvectors \( \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). From here we can proceed in two slightly different ways.

**Method 1** Observe that \( A \) is similar to the diagonal matrix \( D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \), that is, \( S^{-1}AS = D \), where \( S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \) has the corresponding eigenvectors as its columns. Thus \( A = SDS^{-1} \).

We now use this in our differential equation: \( \vec{x}'(t) = SDS^{-1}\vec{x} \). Multiply both sides by \( S^{-1} \). Since \( S \) does not depend on \( t \), \( (S^{-1}\vec{x}(t))' = DS^{-1}\vec{x} \). This is simpler to use if we let \( \vec{y}(t) = S^{-1}\vec{x} \). Then the differential equation becomes

\[
\frac{d\vec{y}(t)}{dt} = D\vec{y}(t),
\]

that is,

\[
\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 3y_1(t) \\ 5y_2(t) \end{pmatrix}.
\]

These are uncoupled differential equations, \( y_1' = 3y_1 \), \( y_2' = 5y_2 \), that one can solve immediately giving

\[
y_1(t) = ae^{3t}, \quad y_2(t) = be^{5t},
\]

for any constants \( a \) and \( b \).

It remains to return to restate this in terms of \( \vec{x}(t) \)

\[
\vec{x}(t) = S\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} ae^{3t} \\ be^{5t} \end{pmatrix} = \begin{pmatrix} ae^{3t} + be^{5t} \\ -ae^{3t} + be^{5t} \end{pmatrix}
\]

We use the initial condition to determine the constants \( a \) and \( b \).

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} a + b \\ -a + b \end{pmatrix}.
\]

Thus \( a = b = 1/2 \). Therefore

\[
\vec{x}(t) = \frac{1}{2} \begin{pmatrix} e^{3t} + e^{5t} \\ -e^{3t} + e^{5t} \end{pmatrix}.
\]
Method 2 Since the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are a basis for $\mathbb{R}^2$, given any $\vec{x}(t)$, there are functions $y_1(t)$ and $y_2(t)$ so that
$$\vec{x}(t) = y_1(t)\vec{v}_1 + y_2(t)\vec{v}_2.$$  \hspace{1cm} (2)

We now plug this in the differential equation $\vec{x}' = A\vec{x}$. The left side becomes
$$\vec{x}'(t) = y_1'(t)\vec{v}_1 + y_2'(t)\vec{v}_2,$$
and the more interesting right side becomes
$$A\vec{x} = 3y_1\vec{v}_1 + 5y_2\vec{v}_2.$$
Comparing the coefficients of $\vec{v}_1$ and $\vec{v}_2$ in the last two equations we conclude that
$$y_1' = 3y_1 \quad \text{and} \quad y_2' = 5y_2.$$
Their solutions are
$$y_1(t) = ae^{3t} \quad \text{and} \quad y_2(t) = be^{5t}$$
for any constants $a$ and $b$. Using this in equation (2) we find
$$\vec{x}(t) = ae^{3t}\left( \begin{array}{c} 1 \\ -1 \end{array} \right) + be^{5t}\left( \begin{array}{c} 1 \\ 1 \end{array} \right).$$
Finally, use the initial condition to determine $a$ and $b$:
$$\left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \vec{x}(0) = a \left( \begin{array}{c} 1 \\ -1 \end{array} \right) + b \left( \begin{array}{c} 1 \\ 1 \end{array} \right).$$
This gives $a = b = 1/2$. Therefore
$$\vec{x}(t) = \frac{1}{2}\left[ e^{3t}\left( \begin{array}{c} 1 \\ -1 \end{array} \right) + e^{5t}\left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right].$$

Example 2 Let $A := \left( \begin{array}{cc} 4 & 1 \\ 1 & 4 \end{array} \right)$ and $\vec{x}(t) = \left( \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right)$. Solve
\begin{align*}
\frac{d^2\vec{x}}{dt^2} &= A\vec{x} \quad \text{with} \quad \vec{x}(0) = \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \quad \text{and} \quad \vec{x}'(0) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right). \hspace{1cm} (3)
\end{align*}

Solution: This is the same as equation (1) except that here we have a second derivative. Both of the methods used in Example 1 work here with essentially no change. We’ll use Method 2.

Since the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of $A$ are a basis for $\mathbb{R}^2$, given any $\vec{x}(t)$, there are functions $y_1(t)$ and $y_2(t)$ so that
$$\vec{x}(t) = y_1(t)\vec{v}_1 + y_2(t)\vec{v}_2.$$  \hspace{1cm} (4)

We now plug this in the differential equation $\vec{x}'' = A\vec{x}$. The left side becomes
$$\vec{x}''(t) = y_1''(t)\vec{v}_1 + y_2''(t)\vec{v}_2,$$
and the more interesting right side becomes

\[ \mathbf{A} \vec{x} = 3y_1 \vec{v}_1 + 5y_2 \vec{v}_2. \]

Comparing the coefficients of \( \vec{v}_1 \) and \( \vec{v}_2 \) in the last two equations we conclude that

\[ y''_1 = 3y_1 \quad \text{and} \quad y''_2 = 5y_2. \]

Both of these equations have the form \( u'' = k^2 u \) whose general solution is

\[ u(t) = c_1 e^{kt} + c_2 e^{-kt}. \]

Thus

\[ y_1(t) = a e^{\sqrt{3} t} + b e^{-\sqrt{3} t} \]
\[ y_2(t) = c e^{\sqrt{5} t} + d e^{-\sqrt{5} t} \]

for any choice of the constants \( a, b, c \) and \( d \). Plug this into equation (4) to find

\[ \vec{x}(t) = (ae^{\sqrt{3} t} + be^{-\sqrt{3} t}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (ce^{\sqrt{5} t} + de^{-\sqrt{5} t}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

We use the initial conditions to determine the constants \( a, b, c \) and \( d \):

\[ \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \vec{x}(0) = (a + b) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (c + d) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]

and

\[ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{x}'(0) = (a - b) \sqrt{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (c - d) \sqrt{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

Therefore, by a routine computation, \( a = b = c = d = \frac{1}{2} \) so

\[ \vec{x}(t) = \frac{1}{2} (e^{\sqrt{3} t} + e^{-\sqrt{3} t}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2} (e^{\sqrt{5} t} + e^{-\sqrt{5} t}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ = \cosh(\sqrt{3} t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \cosh(\sqrt{5} t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ = \begin{pmatrix} \cosh(\sqrt{3} t) + \cosh(\sqrt{5} t) \\ -\cosh(\sqrt{3} t) + \cosh(\sqrt{5} t) \end{pmatrix} \]

Note that any of the last three lines are valid formulas for the solution \( \vec{x}(t) \), your preference depends on what you will next do with the solution.