ODE: Complex Eigenvalues

Say you want to solve the vector differential equation

\[ X'(t) = AX, \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1} \]

If the eigenvalues of \( A \) (and hence the eigenvectors) are real, one has an idea how to proceed. However if the eigenvalues are complex, it is less obvious how to find the real solutions. Because we are interested in a real solution, we need a strategy to untangle this. We examine the case where \( A \) has complex eigenvalues \( \lambda_1 = \lambda \) and \( \lambda_2 = \bar{\lambda} \) with corresponding complex eigenvectors \( W_1 = W \) and \( W_2 = \overline{W} \).

The key observation is that if \( X(t) \) is a complex solution, split \( X \) in its real and imaginary parts, say \( X(t) = U(t) + iV(t) \), where \( U(t) \) and \( V(t) \) are both real vectors. Then

\[
\frac{dX}{dt} = \frac{dU}{dt} + i\frac{dV}{dt} \quad \text{and} \quad AX = AU + iAV.
\]

Thus \( U(t) \) and \( V(t) \) are both real solutions of equation (1).

We apply this observation to our example. For the moment we ignore that in our special case \( \lambda_2 \) and \( W_2 \) are conjugates of \( \lambda_1 \) and \( W_1 \).

Since these \( W_1 \) and \( W_2 \) are linearly independent, we seek any solution as a linear combination,

\[ X(t) = z_1(t)W_1 + z_2(t)W_2. \tag{2} \]

Clearly

\[ X'(t) = z_1'(t)W_1 + z_2'(t)W_2 \]

and

\[ AX = z_1(t)AW_1 + z_2(t)AW_2 = z_1(t)\lambda_1W_1 + z_2(t)\lambda_2W_2. \]

Since we want \( X' = Ax \), comparing the right sides of the equations above we find

\[ z_1' = \lambda_1 z_1 \quad \text{and} \quad z_2' = \lambda_2 z_2. \]
These equations are uncoupled and easy to solve. Their solutions are

\[ z_1(t) = c_1 e^{\lambda_1 t} \quad \text{and} \quad z_2(t) = c_2 e^{\lambda_2 t} \]

for any constants \( c_1 \) and \( c_2 \). Using equation (2) we see that for any constants \( c_1 \) and \( c_2 \)

\[ X(t) = c_1 e^{\lambda_1 t} W_1 + c_2 e^{\lambda_2 t} W_2 \]

is a solution of equation (1). One often calls the \( e^{\lambda_j t} W_j \) eigensolutions of the differential equation.

We apply this to an example of our complex case where \( \lambda_2 \) and \( W_2 \) are complex conjugates of \( \lambda_1 = \lambda \) and \( W_1 = W \) and conclude that

\[ X(t) = c_1 e^\lambda W + c_2 e^\bar{\lambda} \bar{W}. \quad (3) \]

By the key observation made near the top of this page, the real and imaginary parts of the complex eigensolution \( e^{\lambda t} W \) will be the desired real solutions.

**Example** Let \( A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \). By a routine computation the eigenvalues are \( \lambda_1 = \lambda = 3 + i \) and \( \lambda_2 = \bar{\lambda} = 3 - i \) with corresponding eigenvectors \( W = W_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \) and \( W_2 = \bar{W} = \begin{pmatrix} 1 \\ i \end{pmatrix} \). From equation (3) we know that \( e^{\lambda t} W \) is a solution. By Euler’s formula \( e^{x+iy} = e^x (\cos y + i \sin y) \) and then taking real and imaginary parts we find

\[ e^\lambda W = e^{(3+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (e^{3t} \cos t + ie^{3t} \sin t) \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} e^{3t} \cos t \\ e^{3t} \sin t \end{pmatrix} + i \begin{pmatrix} e^{3t} \sin t \\ -e^{3t} \cos t \end{pmatrix}. \]

The last two vectors on the right are the two real solutions we were seeking. Thus, with this \( A \), the most general real solution of \( X' = AX \) is

\[ X(t) = C_1 \begin{pmatrix} e^{3t} \cos t \\ e^{3t} \sin t \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \sin t \\ -e^{3t} \cos t \end{pmatrix} \]
with any real constants $c_1$ and $c_2$.

Note that repeating this process using the second term in equation (3) does not give us anything new because, except for the constants $c_1$ and $c_2$, this term is the complex conjugate of the first term.

The identical procedure works for the more general case of equation (3) and shows that the most general real solution there is by taking any linear combination of the real and imaginary parts of $e^{tW}$.

The next example (taken from *Applied Linear Algebra* by Olver and Shakiban) uses the identical ideas.

**Example**  Solve the initial value problem

\[
\begin{align*}
x'_1 &= x_1 + 2x_2 & & x_1(0) = 2 \\
x'_2 &= x_2 - 2x_3 & & x_2(0) = -1. \\
x'_3 &= 2x_1 + 2x_2 - x_3 & & x_3(0) = -2
\end{align*}
\]

The coefficient matrix is

\[
A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}.
\]

Its eigenvalues and corresponding eigenvectors are

\[
\begin{align*}
\lambda_1 &= -1, & \lambda_2 &= 1+2i, & \lambda_3 &= 1-2i, \\
W_1 &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, & W_2 &= \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}, & W_3 &= \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}.
\end{align*}
\]

The corresponding eigensolutions $Y_j(t) = e^{\lambda_j t}W_j$ are

\[
\begin{align*}
Y_1(t) &= e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, & Y_2(t) &= e^{(1+2i)t} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}, & Y_3(t) &= e^{(1-2i)t} \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}.
\end{align*}
\]

As before, notice that $Y_3 = \bar{Y}_2$. One real solution is evidently $Y_1$. To get the other two, we take the real and imaginary parts of $Y_2$ (or
equivalently, of $Y_3$). The process is routine:

$$Y_2(t) = e^{(1+2i)t} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix} = (e^t \cos 2t + ie^t \sin 2t) \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t \cos 2t \\ -e^t \sin 2t \\ e^t \cos 2t \end{pmatrix} + i \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \\ e^t \sin 2t \end{pmatrix}.$$ 

The three real linearly independent solutions are

$$X_1(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} e^t \cos 2t \\ -e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}, \quad X_3(t) = \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \\ e^t \sin 2t \end{pmatrix}.$$ 

and the general real solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t)$$

for any real constants $c_1, c_2, c_3$. We pick these constants to match the initial conditions

$$c_1 X_1(0) + c_2 X_2(0) + c_3 X_3(0) = X(0),$$

that is,

$$\begin{pmatrix} -c_1 + c_2 \\ c_1 + c_3 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.$$ 

One finds that $c_1 = -2$, $c_2 = -0$, $c_3 = 1$. Thus the final solution is

$$X(t) = \begin{pmatrix} 2e^{-t} + e^t \sin 2t \\ -2e^{-t} + e^t \cos 2t \\ -2e^{-t} + e^t \sin 2t \end{pmatrix}.$$ 

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