\[ A : \mathbb{R}^n \to \mathbb{R}^k, \text{ a Linear Map} \]

**Fact 1** The following are equivalent:
- \( A \) is one-to-one.
- \( \ker(A) = 0 \).
- \( \dim \ker(A) = 0 \).
- The equation \( Ax = y \) has at most one solution.
- The columns of \( A \) are linearly independent.
- The rows of \( A \) span \( \mathbb{R}^n \).
- \( A^T \) is onto.

**Fact 2** The following are equivalent:
- \( A \) is onto.
- \( \text{image}(A) = \mathbb{R}^k \).
- \( \dim \text{image}(A) = k \).
- \( \text{rank} (A) = k \).
- The equation \( Ax = y \) has at least one solution.
- The rows of \( A \) are linearly independent.
- The columns of \( A \) span \( \mathbb{R}^k \).
- \( A^T \) is one-to-one.

**Fact 3** If \( n = k \) the following are equivalent:
- \( A \) is invertible.
- Everything in Fact 1.
- Everything in Fact 2.
- For every \( y \) there is exactly one solution of \( Ax = y \)
- \( A \) is bijective (equivalently, \( A \) is an isomorphism).
- 0 is not an eigenvalue of \( A \).
- \( \det(A) \neq 0 \)