Problem Set 1

Due: In class Thursday, Jan. 23. Late papers will be accepted until 1:00 PM Friday.

These problems are intended to be straightforward with not much computation. I warmly suggest that you discuss these problems with others in the class.

Remark: Before class on Tuesday please read all of Chapters 1 and 2 in the text. Essentially it should be a review from Math 240.

1. Let \( A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \). Compute the inverse of \( A \) and of \( A^2 \).

2. Solve all of the following equations. [Note that the left sides of these equations are identical.]
   a). \( 2x + 5y = 5 \)  
   b). \( 2x + 5y = 0 \)  
   c). \( 2x + 5y = 1 \)  
   d). \( 2x + 5y = 2 \)

3. \[ \text{Bretscher, Sec.2.1 #13} \] Finding the inverse of a matrix \( A \) means solving the system of equations \( A\vec{x} = \vec{y} \) for \( \vec{x} \), so \( \vec{x} = A^{-1}\vec{y} \).
   a) Let \( A := \begin{pmatrix} 1 & 2 \\ c & 6 \end{pmatrix} \). With your bare hands (as on page 2 of the textbook – not using anything about determinants) show that \( A \) is invertible if and only if \( c \neq 3 \).
   b) Let \( M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). With your bare hands (not using anything about determinants) show that \( M \) is invertible if and only if \( ad - bc \neq 0 \). [Hint: Treat the cases \( a \neq 0 \) and \( a = 0 \) separately.]

4. Let \( A \) and \( B \) be \( 2 \times 2 \) matrices.
   a) If \( B \) is invertible and \( AB = 0 \), show that \( A = 0 \).
   b) Give an example where \( AB = 0 \) but \( BA \neq 0 \).
   c) Find an example of a \( 2 \times 2 \) matrix with the property that \( A^2 = 0 \) but \( A \neq 0 \).
   d) Find all invertible \( n \times n \) matrices \( A \) with the property \( A^2 = 3A \).

5. \[ \text{Bretscher, Sec.2.3 #19} \] Find all the matrices that commute with \( A := \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \).

6. a) Find a real \( 2 \times 2 \) matrix \( A \) (other than \( A = \pm I \)) such that \( A^2 = I \).
   b) Find a real \( 2 \times 2 \) matrix \( A \) such that \( A^4 = I \) but \( A^2 \neq I \).

7. Let \( L, M, \) and \( P \) be linear maps from the \((x_1, x_2)\) plane to the \((y_1, y_2)\) plane:
   \( L \) is rotation by 90 degrees counterclockwise.
   \( M \) is reflection across the line \( x_1 = x_2 \).
   \( N\vec{v} := -\vec{v} \) for any vector \( \vec{v} \in \mathbb{R}^2 \).
a) Find matrices representing each of the linear maps $L$, $M$, and $N$.


c) Which pairs of these maps commute?

d) Which of the following identities are correct—and why?

1) $L^2 = N$ 
2) $N^2 = I$ 
3) $L^4 = I$ 
4) $L^5 = L$ 
5) $M^2 = I$ 
6) $M^3 = M$ 
7) $MNM = N$ 
8) $N MN = L$

8. a) Find a $2 \times 2$ matrix that rotates the plane by $+45$ degrees ($+45$ degrees means $45$ degrees counterclockwise).

b) Find a $2 \times 2$ matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

c) Find a $2 \times 2$ matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

9. Let $A$ be a matrix, not necessarily square. Say $V$ and $W$ are particular solutions of the equations $AV = Y_1$ and $AW = Y_2$, respectively, while $Z \neq 0$ is a solution of the homogeneous equation $AZ = 0$. Answer the following in terms of $V$, $W$, and $Z$.

a) Find some solution of $AX = 3Y_1$.

b) Find some solution of $AX = -5Y_2$.

c) Find some solution of $AX = 3Y_1 - 5Y_2$.

d) Find another solution (other than $Z$ and 0) of the homogeneous equation $AX = 0$.

e) Find two solutions of $AX = Y_1$.

f) Find another solution of $AX = 3Y_1 - 5Y_2$.

g) If $A$ is a square matrix, then $\det A =$?

h) If $A$ is a square matrix, for any given vector $W$ can one always find at least one solution of $AX = W$? Why?

[Last revised: January 16, 2014]