1. If $A$ is a positive definite matrix, show that $A^k$ is also positive definite for any integer $k$ (including negative integers). [Note that a positive definite matrix is always assumed to be symmetric.]

2. If $A$ is a positive definite matrix, show that for any invertible matrix $C$, the matrix $C^*AC$ is also positive definite.

3. For a linear programming problem, say the feasible set has the constraints

$$x + 2y \geq 6, \quad 2x + y \geq 6, \quad x \geq 0, \quad y \geq 0.$$ 

Sketch this. What are the three “corners” of this set?

4. In a linear programming problem, show that the feasible set constrained by

$$2x + 5y \leq 3, \quad -3x + 8y \leq -5, \quad x \geq 0, \quad y \geq 0$$

is empty.

5. In a certain factory, say there are three machines $M_1$, $M_2$, $M_3$ used in making two products $P_1$, $P_2$. One unit of $P_1$ occupies $M_1$ for 5 minutes, $M_2$ for 3 minutes, and $M_3$ for 4 minutes. The corresponding data for one unit of $P_2$ are: $M_1$ for 1 minute, $M_2$ for 4 minutes, and $M_3$ for 3 minutes.

The net profit per unit of $P_1$ produced is 30 dollars, and for $P_2$ 20 dollars. The question is: what production plan gives the most profit?

Suppose that $x_1$ units of $P_1$ and $x_2$ units of $P_2$ are produced per hour.

a) If you want to maximize the profits, what function $F(x_1, x_2)$ are you maximizing?
b) Since an hour has 60 minutes, the capacities of the three machines each impose a constraint on $x_1$ and $x_2$. What are these three constraints?
c) Draw a graph of the $x_1$-$x_2$ plane showing the constraints and the feasible region (a pentagon).
d) Compute the maximum profit per hour in this example.
6. [Transportation Problem] Suppose that Texas, California, and Alaska each produce a million barrels of oil:

- 800,000 barrels are needed in Chicago at a distance of 1000, 2000, and 3000 miles from the three producers, respectively.
- 2,200,000 barrels are needed in New England 1500, 3000, and 3700 miles away.

If shipments cost one unit for each barrel-mile, what linear programming problem with five equality constraints (if needed, introduce slack variables) must be used to minimize the shipping costs?

7. Let $A : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be linear map (so it is represented by an $n \times k$ matrix).

a) Show that the matrices $A^*A$ and $AA^*$ both have the same non-zero eigenvalues.

b) If $\vec{v}_1$ and $\vec{v}_2$ are orthogonal eigenvectors of $A^*A$ corresponding to non-zero eigenvalues, and if $\vec{u}_j = A\vec{v}_j$, show that the $\vec{u}_j$ are also orthogonal. [Noted later: Alas, this is a repeat of Homework Set 9, #4].

**Bonus Problem**

[Please give this directly to Professor Kazdan]

B-1 Let $P_1, P_2, \ldots, P_k$ be $k$ points (think of them as data) in $\mathbb{R}^2$ and let $S$ be the line

$$S := \{ X \in \mathbb{R}^2 : \langle X, N \rangle = c \},$$

where $N \neq 0$ is a unit vector normal to the line and $c$ is a real constant.

This problem outlines how to find the line that best approximates the data points in the sense that it minimizes the function

$$Q(N, c) := \sum_{j=1}^k \text{distance}(P_j, S)^2.$$

Determining this line means finding $N$ and $c$.

a) Show that for a given point $P$, then

$$\text{distance}(P, S) = |\langle P - X, N \rangle| = |\langle P, N \rangle - c|,$$

where $X$ is any point in $S$.

b) First do the special case where the center of mass $\overline{P} := \frac{1}{k} \sum_{j=1}^k P_j$ is at the origin, so $\overline{P} = 0$. Show that for any $P$, then $\langle P, N \rangle^2 = \langle N, PP^*N \rangle$. Here view $P$ as a column vector so $PP^*$ is a $3 \times 3$ matrix.
Use this to observe that the desired line $S$ is determined by letting $N$ be an eigenvector of the matrix 

$$A := \sum_{j=1}^{k} P_j P_j^T$$

corresponding to it’s lowest eigenvalue. What is $c$ in this case?

c) Reduce the general case to the previous case by letting $V_j = P_j - \overline{P}$.

d) Find the equation of the line $ax + by = c$ that, in the above sense, best fits the data points $(-1, 3), (0, 1), (1, -1), (2, -3)$.