Problem Set 4 Solutions

Due: In class Thursday, Thurs. Feb. 13. Late papers will be accepted until 1:00 PM Friday.

Reminder: Exam 1 is on Tuesday, Feb. 18, 9:00–10:20. No books or calculators but you may always use one 3” × 5” card with handwritten notes on both sides.

For the coming week, please review Chapter 4 Sections 4.1 and 4.2. Also read Sections 5.1 and 5.2 (we will skip the QR Factorization) and the notes http://www.math.upenn.edu/~kazdan/312S13/notes/vectors/vectors10.pdf on Vectors and Least Squares and http://www.math.upenn.edu/~kazdan/312S13/notes/OrthogProj.pdf on Orthogonal Projections. Later we will return in greater detail to the material in Sections 3.4 and 4.3.

1. Find a basis for the linear space of matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with the property that \( a + d = 0 \).

What is the dimension of this space?

Solution

Such matrices are of the form \( \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \) hence any element of this linear space can be written as

\[
\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Hence the three matrices appearing in this decomposition form a basis of this linear space and the dimension is 3.

2. Find a linear map \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) whose kernel is exactly the plane

\[
\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + 2x_2 - x_3 = 0\}.
\]

Solution

Let \( L = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

3. a) [Like Bretscher, Sec. 4.2 #66] Find the kernel of the map \( T : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) \) defined by \( T(u) := u' - 4u \). What is the dimension of the kernel?

b) Repeat this for \( Tu := u'' - 4u \).

Solution

a) The kernel of \( T \) will be the set of solutions of \( u' = 4u \) which is \( \{ce^{4x} | c \in \mathbb{R}\} \). So it has dimension 1.
b) The kernel of $T$ will be the set of solutions of $u'' = 4u$ which is \( \{ae^{2x} + be^{-2x} | a, b \in \mathbb{R} \} \). So the kernel has dimension 2.

4. We want to approximately compute \( \int_0^2 \frac{1}{1+x^2} \, dx \) by partitioning the interval \( 0 \leq x \leq 2 \) into four sub-intervals whose end point are \( x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2 \). of width \( h = x_{i+1} - x_i = 1/2 \). In each sub-interval replace the integrand by a simpler function.

TRAPEZOIDAL RULE: Approximate the function \( f(x) \) in each sub-interval \( [x_i, x_{i+1}] \) by a straight line joining its end points: \((x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))\).

SOLUTION The values of the function \( y_i = f(x_i) = \frac{1}{1+x_i^2} \) at the points \( x_i \) are
\[ y_0 = 1, \ y_1 = 4/5 = 0.8, \ y_2 = 1/2 = 0.5, \ y_3 = 4/13 = 0.3077, \ y_4 = 1/5 = 0.2 \]
Then the straight line between \((x_i, y_i), (x_{i+1}, y_{i+1})\) is given by
\[ y = y_i + \frac{y_{i+1} - y_i}{h}(x - x_i). \]
After a simple calculation
\[ \int_{x_i}^{x_{i+1}} \left[ y_i + \frac{y_{i+1} - y_i}{h}(x - x_i) \right] \, dx = \frac{h}{2} (y_i + y_{i+1}). \]
Hence the approximation is
\[ \int_0^2 \frac{1}{1+x^2} \, dx \approx \sum_{i=0}^{4} \frac{h}{2} (y_i + y_{i+1}) = \frac{1}{4} (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) = 1.10384, \]

SIMPSON’S RULE: This works with two sub-intervals at a time, say \( x_0 \leq x \leq x_1 \) and \( x_1 \leq x \leq x_2 \) and uses a parabola,
\[ p(x) := a + bx + cx^2 \]
that passes through the three points \((x_0, y_0), (x_1, y_1), \text{ and } (x_2, y_2).\) The idea is to approximate the area under the function in the interval \( x_0 \leq x \leq x_2 \) by the area under the parabola.

SOLUTION The computation is simpler if for the moment we let \( x_0 = -h, \ x_1 = 0, \) and \( x_2 = h. \) Then the conditions that \( p(-h) = y_0, \ p(0) = y_1 \) and \( p(h) = y_2 \) become
\[ a - bh + ch^2 = y_0, \ a = y_1, \text{ and } a + bh + ch^2 = y_2. \]
Instead of computing \( a, \ b, \) and \( c \) immediately, we first do the next step of the approximate integration using \( f(x) \approx p(x) \) in the interval \(-h \leq x \leq h.\) Then
\[ \int_{-h}^{h} f(x) \, dx \approx \int_{-h}^{h} (a + bx + cx^2) \, dx = 2ah + \frac{2}{3} ch^3 = \frac{h}{3} (6a + 2ch^2). \]
But from equations (2), \(a = y_1\) and \(2ch^2 = y_0 + y_2 - 2y_1\). Thus

\[
\int_{-h}^{h} f(x) \, dx \approx \frac{h}{3} (6y_1 + y_0 + y_2 - 2y_1) = \frac{h}{3} (y_0 + 4y_1 + y_2).
\]

Note that this formula depends only on the spacing, \(h\) of the \(x_j\) and the values of the corresponding \(y_j\). In particular, we get

\[
\int_{x_0}^{x_2} f(x) \, dx \approx \frac{h}{3} (y_0 + 4y_1 + y_2).
\]

Similarly,

\[
\int_{x_2}^{x_4} f(x) \, dx \approx \frac{h}{3} (y_0 + 4y_1 + y_2).
\]

Adding these we obtain

\[
\int_{x_0}^{x_4} f(x) \, dx \approx \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4).
\]

This is Simpson’s Rule. More generally, Simpson’s Rule works with a pair of adjacent subintervals so there must be an even number of subintervals whose endpoints are \(x_0, x_1, \ldots, x_n\). Then it gives

\[
\int_{x_0}^{x_n} f(x) \, dx \approx \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).
\]

We now apply Simpson’s Rule (3) to our particular problem (the data (1)) and find

\[
\int_{0}^{2} \frac{1}{1 + x^2} \, dx \approx 1.10513.
\]

5. Find a basis for the space \(\mathcal{P}_4\) of polynomials \(p(x)\) degree at most 4 with the properties \(p(1) = 0\) and \(p(3) = 0\). What is the dimension of this space?

**Solution** We write our polynomials as \(p(x) = a_0 + a_1 x + a_2 x^2 + a_3 4x^3 + a_4 x^4\). Then the conditions \(p(1) = 0\) and \(p(3) = 0\) are

\[
a_0 + a_1 + a_2 + a_3 + a_4 = 0, \quad \text{and} \quad a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 = 0.
\]

We use these to solve for \(a_0\) and \(a_1\) in terms of \(a_2\), \(a_3\), and \(a_4\) and then use these in \(p(x)\) to obtain

\[
p(x) = (3 - 4x + x^2)a_2 + (12 - 13x + x^3)a_3 + (39 - 40x + x^4)a_4.
\]

The three polynomials

\[
p_1(x) = 3 - 4x + x^2, \quad p_2(x) = 12 - 13x + x^3, \quad \text{and} \quad p_3(x) = 39 - 40x + x^4
\]
are a basis for this space so its dimension is 3.

Alternate The above computation was unpleasant, and leads us to seek a different basis that is better adapted to this space. It uses the observation that each polynomial $p(x)$ in this space must have the quadratic polynomial $(x - 1)(x - 3)$ as a factor. Thus $p(x)$ must have the form

$$p(x) = (x - 1)(x - 3)(\alpha + \beta x + \gamma x^2)$$

for any numbers $\alpha$, $\beta$, and $\gamma$. This alternate basis consists of the three polynomials

$$q_1(x) = (x - 1)(x - 3), \quad q_2(x) = (x - 1)(x - 3)x, \quad \text{and} \quad q_3(x) = (x - 1)(x - 3)x^2.$$

6. In class we considered the interpolation problem of finding a polynomial of degree $n$ passing through $n + 1$ specified distinct points in the plane. To be definite, take $n = 3$, and say our points are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, and $(x_4, y_4)$. This problem involves $P_3$, and so we could work in the usual basis $\{1, x, x^2, x^3\}$. However, it is easier to use the Lagrange basis. The point of this problem is to see vividly why choosing a basis adapted to the problem may involve much less work.

a) Setup the linear equations you would need to solve to find the polynomial of degree 3 passing through the points $(0, -3)$, $(1, -1)$, $(2, 11)$, and $(-1, -7)$ if you use the usual basis $\{1, x, x^2, x^3\}$. But don’t take time to solve these.

Solution Let $p(x) = a + bx + cx^2 + dx^3$. Then we want:

$$a = -3, \quad a + b + c + d = -1, \quad a + 2b + 4c + 8d = 11, \quad a - b + c - d = -7,$$

or equivalently,

$$b + c + d = 2, \quad 2b + 4c + 8d = 14, \quad -b + c - d = -4,$$

so we have a $3 \times 3$ system to determine the remaining coefficients $b$, $c$, $d$. Not fun.

b) Solve the same problem explicitly using the Lagrange basis.

Solution The Lagrange basis consists of four polynomials $p_i(x)$, $i = 0, \ldots, 3$ with the property that

$$p_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

For instance,

$$p_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x - 1)(x - 2)(x + 1)}{(0 - 1)(0 - 2)(0 + 1)}.$$

Then

$$p(x) = y_0 p_0(x) + y_1 p_1(x) + y_2 p_2(x) + y_3 p_3(x) = -3p_0(x) - 1p_1(x) + 11p_2(x) - 7p_3(x).$$
7. [[Bretscher, Sec. 4.2 #70] Does there exist a polynomial \( f(t) \) of degree at most 4 such that \( f(2) = 3 \), \( f(3) = 5 \), \( f(5) = 7 \), \( f(7) = 11 \), and \( f(11) = 2 \)? If so, how many such polynomials are there? [: Note: This problem only asks if such a polynomial exists. It is not asking you to find it.]

**Solution**  
Polynomial Interpolation suggests that there exists a polynomial \( f(t) \) of degree at most 4 passing through 5 specified distinct points and using the Lagrange basis as in problem 6 part (b) we can define \( f(t) \). Now suppose \( f(t), g(t) \) are such polynomials. Then \( r(t) = f(t) - g(t) \) is a polynomial of degree at most 4 which has 5 roots. That is a contradiction since a polynomial of degree \( n \) has at most \( n \) roots. Hence there is a unique such polynomial.

More formally, define the linear map \( L : P_4 \rightarrow \mathbb{R}^5 \) by the rule
\[
Lf := (f(2), f(3), f(5), f(7), f(11))
\]
The kernel of \( L \) are the quartic polynomials that are zero at the four points \( x = 2, 3, 5, 7, 11 \). But the only quartic polynomial that has four zeroes is the zero polynomial. Thus \( \ker(L) = 0 \). Since \( P_4 \) and \( \mathbb{R}^5 \) both have the same dimension, 5, by the rank-nullity theorem \( L \) is invertible. Thus given any numbers \( y_0, y_1, y_2, y_3 \) and \( y_4 \) there is a unique cubic polynomial \( f(t) \) so that \( f(2) = y_0, f(3) = y_1, f(5) = y_2, f(7) = y_3 \), and \( f(11) = y_4 \).

8. Let \( P_2 \) be the linear space of polynomials of degree at most 2 and \( T : P_2 \rightarrow P_2 \) be the transformation
\[
(T(p))(t) = \frac{1}{t} \int_0^t p(s) \, ds.
\]
For instance, if \( p(t) = 2 + 3t^2 \), then \( T(p) = 2 + t^2 \).

a)  Prove that \( T \) is a linear transformation.

**Solution**  
Linearity of integral give us directly that \( T \) is a linear transformation. Actually if we let \( p(s) = a + bs + cs^2 \) then \((T(p))(t) = a + \frac{b}{2} t + \frac{c}{3} t^2 \).

b)  Find the kernel of \( T \), and find its dimension.

**Solution**  
We want \( p(s) = a + bs + cs^2 \) such that \( a + \frac{b}{2} t + \frac{c}{3} t^2 = 0 \) for all \( t \). Hence \( p(s) = 0 \) and the kernel is trivial namely its dimension is 0.

c)  Find the range (=image) of \( T \), and compute its dimension.

**Solution**  
Then image contains elements \( a + \frac{b}{2} t + \frac{c}{3} t^2 = a + \hat{b} t + \hat{c} t^2 \) for any \( a, \hat{b}, \hat{c} \in \mathbb{R} \) hence \( \text{im}T = P_2 \), namely the dimension of the image is 3.

d)  Verify the dimension of the kernel and the dimension of the image add up to what you would expect.

**Solution**  
Indeed \( \dim(\text{im}T) + \dim(\ker T) = 3 = \dim P_2 \).
e) Using the standard basis \{1, t, t^2\} for \(P_2\), represent the linear transformation \(T\) as a matrix \(A\).

**Solution**  The previous description of \(T\) gives us that \(A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}\).

f) Using your matrix representation from (e), find \(T(p)\) where \(p(t) = t - 2\).

**Solution**  \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/2 \\ 0 \end{pmatrix}
\] hence \(T(p) = -2 + 1/2t\).

The remaining problems are from the Lecture notes on Vectors


9. [p. 8 #5] The origin and the vectors \(X\), \(Y\), and \(X + Y\) define a parallelogram whose diagonals have length \(X + Y\) and \(X - Y\). Prove the parallelogram law

\[\|X + Y\|^2 + \|X - Y\|^2 = 2\|X\|^2 + 2\|Y\|^2;\]

This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

**Solution:**  The standard procedure is to express the norm in terms of the inner product and use the usual algebraic rules for the inner product. Thus

\[\|X + Y\|^2 = \langle X + Y, X + Y \rangle = \langle X, X \rangle + \langle X, Y \rangle + \langle Y, X \rangle + \langle Y, Y \rangle = \langle X, X \rangle + 2\langle X, Y \rangle + \langle Y, Y \rangle,
\]

with a similar formula for \(\|X - Y\|^2\). After easy algebra, the result is clear.

10. [p. 8 #6] (Math 240 Review)

a) Find the distance from the straight line \(L\): \(3x - 4y = 10\) to the origin. [It may help to observe that this line is parallel to the plane \(3x - 4y = 0\), whose normal vector is clearly \(\vec{N} = (3, -4)\).]

**Solution:**  Note that the equation of the parallel line \(L_0\) through the origin is \(3x - 4y = 0\), which we rewrite as \(\langle N, X \rangle = 0\), where \(N := (3, -4)\) and \(X = (x, y)\). Let \(X_0\) be some point on the original line (say \(X_0 = (2, -1)\) – although we won’t need to be specific), so \(\langle N, X_0 \rangle = 10\). Then the desired distance \(D\) is the same as the distance from \(X_0\) to the line \(L_0\): \(\langle N, X \rangle = 0\), through the origin. But the equation for \(L_0\) says the vector \(N\) is perpendicular to the line \(L_0\). Thus the distance \(D\) is the length of the projection of \(X_0\) in the direction of \(N\), that is,

\[D = \frac{\|\langle N, X_0 \rangle\|}{\|N\|} = \frac{10}{5} = 2.\]
b) Find the distance from the plane $ax + by + cz = d$ to the origin (assume the vector $\vec{N} = (a, b, c) \neq 0$).

**Solution:** If $X_0$ is some point on the plane, the equation of this plane is $\langle N, X \rangle = \langle N, X_0 \rangle$. The solution presented in the above special case generalizes immediately to give

$$D = \frac{|\langle N, X_0 \rangle|}{\|N\|} = \frac{|d|}{\|N\|}.$$

11. [p. 8 #8]

a) If $X$ and $Y$ are real vectors, show that

$$\langle X, Y \rangle = \frac{1}{4} \left( \|X + Y\|^2 - \|X - Y\|^2 \right).$$

This formula is the simplest way to recover properties of the inner product from the norm.

**Solution:** The straightforward procedure is the same as in Problem 9: rewrite the norms on the right side of equation (4) in terms of the inner product and expand using algebra.

b) As an application, show that if a square matrix $R$ has the property that it preserves length, so $\|RX\| = \|X\|$ for every vector $X$, then it preserves the inner product, that is, $\langle RX, RY \rangle = \langle X, Y \rangle$ for all vectors $X$ and $Y$.

**Solution:** We know that $\|RZ\| = \|Z\|$ for any vector $Z$. This implies $\|R(X + Y)\| = \|X + Y\|$ for any vectors $X$ and $Y$, and, similarly, $\|R(X - Y)\| = \|X - Y\|$ for any vectors $X$ and $Y$. Consequently, by equation (4) (used twice)

$$4\langle RX, RY \rangle = \|R(X + Y)\|^2 - \|R(X - Y)\|^2$$

$$= \|X + Y\|^2 - \|X - Y\|^2$$

$$= 4\langle X, Y \rangle$$

for all vectors $X$ and $Y$.

12. [p. 9 #10] (Also done in class)

a) If a certain matrix $C$ satisfies $\langle X, CY \rangle = 0$ for all vectors $X$ and $Y$, show that $C = 0$.

**Solution:** Since $X$ can be any vector, let $X = CY$ to show that $\|CY\|^2 = \langle CY, CY \rangle = 0$. Thus $CY = 0$ for all $Y$ so $C = 0$.

b) If the matrices $A$ and $B$ satisfy $\langle X, AY \rangle = \langle X, BY \rangle$ for all vectors $X$ and $Y$, show that $A = B$.

**Solution:** We have

$$0 = \langle X, AY \rangle - \langle X, BY \rangle = \langle X, (AY - BY) \rangle = \langle X, (A - B)Y \rangle$$

for all $X$ and $Y$ so by part (a) with $C := A - B$, we conclude that $A = B$. 

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(a) Give an example of a $3 \times 3$ anti-symmetric matrix (other than the trivial $A = 0$).

\textbf{Solution:} The most general anti-symmetric $3 \times 3$ matrix has the form

$$
\begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{pmatrix}.
$$

(b) If $A$ is any anti-symmetric matrix, show that $\langle X, AX \rangle = 0$ for all vectors $X$.

\textbf{Solution:} The key point is to use the definition of $A^*$ having the property $\langle X, AY \rangle = \langle A^*X, Y \rangle$ for all $X$ and $Y$. This is equivalent to $\langle AX, Y \rangle = \langle X, A^*Y \rangle$ for all $X$ and $Y$. [Using the fact that for a matrix $A^*$ happens to be the transpose often causes extra confusion.] Thus

$$
\langle X, AX \rangle = \langle A^*X, X \rangle = -\langle AX, X \rangle = -\langle X, AX \rangle.
$$

so $2\langle X, AX \rangle = 0$ and we are done: $\langle X, AX \rangle = 0$.

[Last revised: February 17, 2014]