Problem Set 6
Due: In class Thurs, March 6. Late papers will be accepted until 1:00 PM Friday.

1. [Bretscher, Sec. 5.1 #26] Find the orthogonal projection $P_S$ of $\vec{x} := \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix}$ into
the subspace $S$ of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

Solution: We are fortunate that the vectors $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. We want to find constants $a$ and $b$ so that

$$\vec{x} = a\vec{v}_1 + b\vec{v}_2 + \vec{w},$$

where $\vec{w}$ is orthogonal to $S$. Then the desired projection will be $P_S\vec{x} = a\vec{v}_1 + b\vec{v}_2$. To find the scalars $a$ and $b$, take the inner product of (1) with $\vec{v}_1$ and then $\vec{v}_2$ we find

$$\langle \vec{x}, \vec{v}_1 \rangle = a\|\vec{v}_1\|^2$$ and $$\langle \vec{x}, \vec{v}_2 \rangle = b\|\vec{v}_2\|^2.$$

Using the particular vectors in this problem, $a = 11$ and $b = -1$. Thus

$$P_S\vec{x} = 11\vec{v}_1 - \vec{v}_2 = \begin{pmatrix} 19 \\ 39 \\ 64 \end{pmatrix}$$

2. [Bretscher, Sec. 5.4 #2] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$. Find a basis for $\text{ker} A^*$. Draw a sketch illustrating the formula $(\text{im} A)^\perp = \text{ker} A^*$ in this case.

Solution: We need to solve $A^*\vec{x} = 0$, namely:

$$x_1 + x_2 + x_3 = 0, \quad x_1 + 2x_2 + 3x_3 = 0$$

Hence we obtain that $x_2 = -2x_3$ and $x_1 = x_3$ so $\vec{v} := (1, -2, 1)$ is a basis for $\text{ker} A^*$. 

Plane is $\text{im}(A)$.
3. [Bretscher, Sec. 5.4 #16] Let \( A : \mathbb{R}^k \to \mathbb{R}^n \) be an \( n \times k \) matrix. Show that 
\[ \text{rank } A^* = \text{rank } A, \quad \text{that is, } \dim(\text{image } A^*) = \dim(\text{image } A) \]

**Solution:** We will use the two formulas 
\[ (\text{image } A)^\perp = \ker A^* \quad \text{and} \quad \dim(\ker A^*) + \dim(\text{image } A^*) = n \]
[or the equivalent formulas interchanging the roles of \( A \) and \( A^* \)]. Since 
\[ \dim(\text{image } A) + \dim(\text{image } A)^\perp = n, \]
the first formula implies 
\[ n - \dim(\text{image } A) = \dim(\ker A^*), \]
while the second implies 
\[ n - \dim(\text{image } A^*) = \dim(\ker A^*) \]
The result is now clear.

4. [Bretscher, Sec. 5.2 #32] Find an orthonormal basis for the plane \( x_1 + x_2 + x_3 = 0 \).

**Solution:** Pick any point in the plane, say \( \vec{v}_1 = (1, -1, 0) \). This will be the first vector in our orthogonal basis. We use the Gram-Schmidt process to extend this to an orthogonal basis for the plane.

Pick any other point in the plane, say \( \vec{w}_1 := (1, 0, -1) \). Write it as \( \vec{w}_1 = a\vec{v}_1 + \vec{z} \), where \( \vec{z} \) is perpendicular to \( \vec{v}_1 \). Note that, although unknown, \( \vec{z} \) will also be in the plane since it will be a linear combination of \( \vec{v}_1 \) and \( \vec{w} \), both of which are in the plane. As usual, by taking the inner product of both sides of \( \vec{w}_1 = a\vec{v}_1 + \vec{z} \) with \( \vec{v}_1 \), we find 
\[ a = \langle \vec{w}_1, \vec{v}_1 \rangle / \| \vec{v}_1 \|^2 \]
Thus 
\[ \vec{z} = \vec{w}_1 - a\vec{v}_1 = \left( \frac{1}{2}, \frac{1}{2}, -1 \right) \]
is in the plane and orthogonal to \( \vec{v}_1 \). The vectors \( \vec{v}_1 \) and \( \vec{z} \) are an orthogonal basis for this plane. To get an orthonormal basis we just make these into unit vectors
\[ \vec{u}_1 := \frac{\vec{v}_1}{\| \vec{v}_1 \|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{u}_2 := \frac{\vec{z}}{\| \vec{z} \|} = \frac{1}{\sqrt{3/2}} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} \]
5. [Bretscher, Sec. 5.3 #10] Consider the space $P_2$ of real polynomials of degree at most 2 with the inner product
\[ \langle f, g \rangle = \frac{1}{2} \int_{-1}^{1} f(t)g(t) \, dt. \]
Find an orthonormal basis for all the functions in $P_2$ that are orthogonal to $f(t) = t$.

**Solution:** We have that $\{1, t, t^2\}$ is a basis for $P_2$. The orthogonal complement of $t$ has dimension 2. Because $t$ is an odd polynomial and both 1 and $t^2$ are even, the elements in $P_2$ that are orthogonal to $t$ are the polynomials of the form $p(t) = a \cdot 1 + bt^2$.

We want an orthonormal basis for this.

First an orthogonal basis. Let $p_1(t) = 1$. This will be the first element of our orthogonal basis. For the second we write $t^2 = a \cdot 1 + p_2(t)$, where $p_2(t)$ is orthogonal to $p_1(t)$.

As usual, take the inner product of both sides of this with $p_1(t)$ to find $\langle t^2, 1 \rangle = a \langle 1, 1 \rangle + \langle p_2, 1 \rangle$. Since $\|1\| = 1$ and we want $p_2 \perp 1$, this means $\langle t^2, 1 \rangle = a \|1\|^2 + 0 = a$. But
\[ \langle t^2, 1 \rangle = \frac{1}{2} \int_{-1}^{1} t^2 \cdot 1 \, dt = \frac{1}{3}. \]
Thus $a = 1/3$ and hence $p_2(t) = t^2 - 1/3$.

To make $p_1, p_2$ into an orthonormal basis we compute
\[ \|p_2\|^2 = \frac{1}{2} \int_{-1}^{1} \left( t^2 - \frac{1}{3} \right)^2 \, dt = \frac{4}{45}. \]

An orthonormal basis of the polynomials in $P_2$ that are orthogonal to $t$ is thus
\[ e_1(t) = 1, \quad e_2(t) = \frac{t^2 - \frac{1}{3}}{\sqrt{\frac{4}{45}}} = \frac{\sqrt{5}}{2} (3t^2 - 1). \]

6. [Bretscher, Sec. 5.3 #16] Consider the space $P_1$ with the inner product
\[ \langle f, g \rangle = \int_{0}^{1} f(t)g(t) \, dt. \]

a) Find an orthonormal basis for this space. [Suggestion: Let $e_1(t) = 1$ and pick $e_2(t) = a + bt$ to be orthogonal to $e_1$.]

**Solution:** We let $e_1(t) = 1$ (it already has length 1). For $e_2$ to be orthogonal to $e_1$ we need $e_2(t) = c(t - 1/2)$ for some constant $c$. Since $\int_{0}^{1} (t - 1/2)^2 \, dt = 1/12$, then
\[ e_2(t) = \sqrt{12} (t - 1/2) = \sqrt{3} (2t - 1). \]
b) Find the linear polynomial \( g(t) = a + bt \) that best approximates the polynomial \( f(t) = t^2 \). Thus, one wants to pick \( g(t) \) so that \( \| f - g \| \) is as small as possible.

[Question: In an inner product space \( V \), if you have a subspace \( S \subset V \) and a vector \( \vec{y} \in V \), how can you find the vector in \( S \) that is closest to \( \vec{y} \)?]

**Solution:** Use the orthogonal projection on \( S \). Since \( \langle t^2, 1 \rangle = 1/3 \) and \( \langle t^2, \sqrt{3}(2t - 1) \rangle = \frac{\sqrt{3}}{6} \), then

\[
g(t) = \text{proj}_S f(t) = \sum_i \langle e_i(t), f(t) \rangle e_i(t) = 1/3 + \frac{\sqrt{3}}{6} \sqrt{3}(2t - 1) = -1/6 + t.
\]

7. Let \( f(x) := \begin{cases} 0 & \text{if } -\pi \leq x \leq -\pi/2 \\ 1 & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 \leq x \leq \pi \end{cases} \) and define \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx \). Find the Fourier Series of \( f(x) \).

**Solution:** Using the formulas for the coefficients we have: \( a_0 = \sqrt{\pi/2} \) and for \( n \geq 1 \),

\[
a_n = \int_{-\pi/2}^{\pi/2} \frac{\cos nx}{\sqrt{\pi}} dx = \frac{2}{n\sqrt{\pi}} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \frac{2}{n\sqrt{\pi}} & \text{if } n = 1, 5, 9, \ldots, \\ -\frac{2}{n\sqrt{\pi}} & \text{if } n = 3, 7, 11, \ldots \end{cases}
\]

Similarly, since \( \sin nx \) is an odd function,

\[
b_n = \frac{1}{\sqrt{\pi}} \int_{-\pi/2}^{\pi/2} \sin nx dx = 0,
\]

Hence,

\[
f(x) = \frac{1}{2} + \frac{2}{\pi} \left[ \cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots \right]
\]

8. [Bretscher, Sec. 5.1 #37] Consider a plane \( V \) in \( \mathbb{R}^3 \) with orthonormal basis \( \vec{u}_1 \) and \( \vec{u}_2 \). Let \( \vec{x} \) be a vector in \( \mathbb{R}^3 \). Find a formula for the orthogonal reflection \( R_V \vec{x} \) of \( \vec{x} \) across the plane \( V \). Your answer will involve \( P_V \vec{x} \), the orthogonal projection of \( \vec{x} \) into the plane \( V \). [Suggestion: Use that \( (I - P_V)\vec{x} \) is the component of \( \vec{x} \) that is orthogonal to \( V \). In a reflection, this is the part of \( \vec{x} \) that is flipped.]

**Solution:** The key is a picture (first try it in \( \mathbb{R}^2 \) where \( V \) is a line through the origin). Let \( P_V \vec{x} \) be the orthogonal projection of \( \vec{x} \) into the plane \( V \). Then \( \vec{w} := P_{V \perp} \vec{x} = \vec{x} - P_V \vec{x} \) is the projection of \( \vec{x} \) orthogonal to \( V \). From the picture, to get the reflection, replace \( \vec{w} \) by \( -\vec{w} \)

Thus, since \( \vec{x} = P_V \vec{x} + \vec{w} \), then

\[
R_V \vec{x} = P_V \vec{x} - \vec{w} = P_V \vec{x} - (\vec{x} - P_V \vec{x}) = 2P_V \vec{x} - \vec{x}.
\]
In summary, orthogonal projections and reflections for a subspace \( V \) are related by the simple formula \( R_V = 2P_V - I \).

Note that if you know an orthonormal basis for \( V \), the orthogonal projection, \( P_V \), is easy to compute. All of this is very general. In this problem \( \tilde{u}_1 \) and \( \tilde{u}_2 \) are an orthonormal basis for the subspace \( V \), so

\[
P_V \tilde{x} = \langle \tilde{x}, \tilde{u}_1 \rangle \tilde{u}_1 + \langle \tilde{x}, \tilde{u}_2 \rangle \tilde{u}_2.
\]

Consequently,

\[
R_V \tilde{x} = 2( \langle \tilde{x}, \tilde{u}_1 \rangle \tilde{u}_1 + \langle \tilde{x}, \tilde{u}_2 \rangle \tilde{u}_2 ) - \tilde{x}.
\]

9. Let \( V \) be a linear space with an inner product and \( P : V \rightarrow V \) a linear map. \( P \) is called a projection if \( P^2 = P \). Let \( Q := I - P \).

a) Show that \( Q^2 = Q \), so \( Q \) is also a projection.

Show that the image of \( P \) is the kernel of \( Q \).

Solution: \( Q^2 = I - PI - IP + P^2 = I - P - P + P = I - P = Q \).

We need to show that \( \text{im } P \subset \ker Q \) and \( \ker Q \subset \text{im } P \). Say \( x \in \text{im } P \), then \( x = Py \) for some \( y \). Thus

\[
(I - P)x = Ix - Px = Ix - P^2y = Ix - Py = x - x = 0.
\]

Conversely, say \( y \in \ker Q \), then \( y = Iy = Py \) so \( y \in \text{im } P \).

b) A projection \( P \) is called an orthogonal projection if the image of \( P \) is orthogonal to the kernel of \( P \). If \( P = P^* \), show that \( P \) is an orthogonal projection.

Solution: Let \( x \in \ker Q = \text{im } P \) and \( y \in \ker P \). Since \( x = Px \) and \( Py = 0 \), then \( \langle x, y \rangle = \langle Px, y \rangle = \langle x, P^*y \rangle = \langle x, Py \rangle = 0 \).

c) Conversely, if \( P \) is an orthogonal projection, show that \( P = P^* \).

Solution: We will show that \( \langle Px, y \rangle = \langle x, Py \rangle \) for all \( x \) and \( y \). Write \( x = Px + (I - P)x = x_1 + x_2 \). Note that \( x_1 \in \text{im } (P) \) and \( x_2 \in \ker (P) \). Similarly
write \( y = Py + (I - P)y = y_1 + y_2 \). By assumption the image and kernel of \( P \) are orthogonal, so \( x_1 \) and \( y_2 \) are orthogonal, as are \( x_2 \) and \( y_1 \). The following computation completes the proof.

\[
\langle Px, y \rangle = \langle x_1, y_1 + y_2 \rangle = \langle x_1, y_1 \rangle \quad \text{and} \quad \langle x, Py \rangle = \langle x_1 + x_2, y_1 \rangle = \langle x_1, y_1 \rangle.
\]

**ALTERNATE:** Since the image and kernel of \( P \) are orthogonal, then \( \langle (I - P)x, Py \rangle = 0 \) for all \( x \) and \( y \). Thus,

\[
\langle x, Py \rangle = \langle Px, Py \rangle = \langle x, P^*Py \rangle
\]

for all \( x \) and \( y \). This implies that \( P = P^*P \). Since \( P^*P \) is self-adjoint, this shows that \( P \) is self-adjoint.

10. Let \( A \) be a real matrix, not necessarily square.
   a) If \( A \) is onto, show that \( A^* \) is one-to-one.
      **Solution:** Since \( \text{im} A^\perp = \ker A^* \), thus \( \ker A = 0 \).
   b) If \( A \) is one-to-one, show that \( A^* \) is onto.
      **Solution:** Similarly, \( \text{im} A^{\perp} = \ker A \).

11. Let \( A \) be a real matrix, not necessarily square.
   a) Show that both \( A^*A \) and \( AA^* \) are self-adjoint.
      **Solution:** Using \( (AB)^* = B^*A^* \) and \( (A^*)^* = A \), this is easy.
      The example \( A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) is illuminating. Here

      \[
      A^*A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad AA^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
      \]

   b) Show that \( \ker A = \ker A^*A \). \([\text{HINT:} \text{ Show separately that } \ker A \subseteq \ker A^*A \text{ and } \ker A \supseteq \ker A^*A. \text{ The identity } \langle \bar{x}, A^*A\bar{x} \rangle = \langle A\bar{x}, A\bar{x} \rangle \text{ is useful.}]\)
      **Solution:** If \( \bar{x} \in \ker A \), then \( A\bar{x} = 0 \) so \( A^*A\bar{x} = A^*0 = 0 \). Thus \( \bar{x} \in \ker A^*A \).
      In other words, \( \ker A \subseteq \ker A^*A \).
      Conversely, if \( \bar{x} \in \ker A^*A \), then \( A^*A\bar{x} = 0 \) so

      \[
      0 = \langle \bar{x}, A^*A\bar{x} \rangle = \langle A\bar{x}, A\bar{x} \rangle = \|A\bar{x}\|^2.
      \]
      Consequently \( A\bar{x} = 0 \), that is, \( \bar{x} \in \ker A \). This proves that \( \ker A^*A \subseteq \ker A \).
   c) If \( A \) is one-to-one, show that \( A^*A \) is invertible
      **Solution:** From part (b) the square matrix \( A^*A \) is 1-1, hence it is invertible.
d) If $A$ is onto, show that $AA^*$ is invertible.

Solution: From exercise 10, part (a) we have that $A^*$ is 1-1. Therefore as in part (c), $AA^*$ is 1-1 so the square matrix $AA^*$ is invertible.

12. [This question is now a bonus question (see below).]

**Quadratic Polynomials Using Inner Products**

If $A$ is a real symmetric matrix (so it is self-adjoint), then $Q(x) := \langle x, Ax \rangle$ is a quadratic polynomial. Given a quadratic polynomial, it is easy to find the (unique) symmetric symmetric matrix $A$. Here is an example. Say $Q(x) := 3x_1^2 - 8x_1x_2 - 5x_2^2$. To find $A$, note that $-8x_1x_2 = -4x_1x_2 - 4x_2x_2$ so we can rewrite $Q$ as

$$Q(x) := 3x_1^2 - 4x_1x_2 - 4x_2x_1 - 5x_2^2.$$ 

If we let

$$A := \begin{pmatrix} 3 & -4 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[Note $A$ is a symmetric matrix],

then it is easy to verify that $Q(x) = \langle x, Ax \rangle$. In the remaining problems we will use this to help work with quadratic polynomials.

13. In each of these find a $3 \times 3$ symmetric matrix $A$ so that $Q(x) = \langle x, Ax \rangle$.

a) $Q(x) := 3x_1^2 - 8x_1x_2 - 5x_2^2 + x_3^2$.

Solution: $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $Q(x) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3 + x_3^2$.

Solution: $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -5 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}$

c) $Q(x) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3$.

Solution: $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -5 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$

14. [Lower order terms and Completing the Square] Which is simpler:

$$z = x_1^2 + 4x_2^2 - 2x_1 + 4x_2 + 2 \quad \text{or} \quad z = y_1^2 + 4y_2^2 ?$$

If we let $y_1 = x_1 - 1$ and $y_2 = x_2 + 1/2$, they are essentially the same. All we did was translate the origin to $(1, -1/2)$. 

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The point of this problem is to generalize this to quadratic polynomials in several variables. Let

\[ Q(\vec{x}) = \sum a_{ij} x_i x_j + 2 \sum b_i x_i + c \]

be a real quadratic polynomial so \( \vec{x} = (x_1, \ldots, x_n) \), \( \vec{b} = (b_1, \ldots, b_n) \) are real vectors and \( A = (a_{ij}) \) is a real symmetric \( n \times n \) matrix.

In the case \( n = 1 \), \( Q(x) = ax^2 + 2bx + c \) which is clearly simpler in the special case \( b = 0 \). In this case, if \( a \neq 0 \), by completing the square we find

\[ Q(x) = a(x + b/a)^2 + c - 2b^2/a = ay^2 + \gamma, \]

where we let \( y = x - b/a \) and \( \gamma = c - b^2/a \). Thus, by translating the origin: \( x = y + b/a \) we can eliminate the linear term in the quadratic polynomial – so it becomes simpler.

a) Similarly, for any dimension \( n \), if \( A \) is invertible, using the above as a model, show there is a change of variables \( \vec{y} := \vec{x} - \vec{v} \) (this is a translation by the vector \( \vec{v} \)) so that in the new \( \vec{y} \) variables \( Q \) has the form

\[ \hat{Q}(\vec{y}) := Q(\vec{y} + \vec{v}) = \langle \vec{y}, A\vec{y} \rangle + \gamma \quad \text{that is}, \quad \hat{Q}(\vec{y}) = \sum a_{ij} y_i y_j + \gamma, \]

where \( \gamma \) involves \( A \), \( b \), and \( c \) – but no terms that are linear in \( \vec{y} \). [In the case \( n = 1 \), which you should try first, this means using a change of variables \( y = x - v \) to change the polynomial \( ax^2 + 2bx + c \) to the simpler \( ay^2 + \gamma \).]

Solutions: First the case \( n = 1 \) again. Then \( Q(x) = Ax^2 + 2bx + c \) so

\[ Q(x) = Q(y + v) = A(y + v)^2 + 2b(y + v) + c \]

\[ = Ay^2 + (2Av + 2b)y + Av^2 + 2bv + c. \]

To kill the linear term, pick \( v \) so that \( 2Av + 2b = 0 \), that is, \( v = -b/A \). Then \( Q(x) = Ay^2 + \gamma \), where

\[ \gamma = Ab^2/A^2 - 2b^2/A + c = -b^2/A + c. \]

Next, the case of arbitrary \( n \). It should now feel routine. We are trying the change of variables \( \vec{x} := \vec{y} - \vec{v} \) with the thought of picking \( \vec{v} \) to simplify the result. The following should be a straightforward computation (the third line uses \( A = A^* \)):

\[ Q(\vec{x}) = Q(\vec{y} + \vec{v}) = \langle \vec{y} + \vec{v}, A(\vec{y} + \vec{v}) \rangle + \langle \vec{b}, \vec{y} + \vec{v} \rangle + c \]

\[ = \langle \vec{y}, A\vec{y} \rangle + \langle \vec{y}, A\vec{v} \rangle + \langle \vec{v}, A\vec{y} \rangle + \langle \vec{v}, A\vec{v} \rangle + 2\langle \vec{b}, \vec{y} \rangle + 2\langle \vec{b}, \vec{v} \rangle + c \]

\[ = \langle \vec{y}, A\vec{y} \rangle + 2A\vec{v} + 2\vec{b}, \vec{y} \rangle + \langle \vec{v}, A\vec{v} \rangle + 2\langle \vec{b}, \vec{v} \rangle + c. \]
The term that is linear in \( y \) will vanish if we pick \( \bar{v} \) so that \( 2A\bar{v} + 2\bar{b} = 0 \), that is, \( \bar{v} = -A^{-1}\bar{b} \). Then

\[
Q(\bar{x}) = \langle \bar{y}, A\bar{y} \rangle + \gamma
\]

where

\[
\gamma = \langle A^{-1}\bar{b}, \bar{b} \rangle - 2\langle \bar{b}, A^{-1}\bar{b} \rangle + c = -\langle \bar{b}, A^{-1}\bar{b} \rangle + c.
\]

This agrees with what we found in the special case \( n = 1 \).

b) As an example, apply this to \( Q(\bar{x}) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4 \).

**Solution:** Here \( Q(\bar{x}) = \langle \bar{x}, A\bar{x} \rangle + 2\langle \bar{b}, \bar{x} \rangle + c \), where \( A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \), \( \bar{b} = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix} \), and \( c = -4 \). Thus \( A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \) so \( \bar{v} = -A^{-1}\bar{b} = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix} \).

15. For \( \bar{x} \in \mathbb{R}^n \) let \( Q(\bar{x}) := \langle \bar{x}, A\bar{x} \rangle \), where \( A \) is a real symmetric matrix. We say that \( A \) is **positive definite** if \( Q(\bar{x}) > 0 \) for all \( \bar{x} \neq 0 \), **negative definite** if \( Q(\bar{x}) < 0 \) for all \( \bar{x} \neq 0 \), and **indefinite** if \( Q(\bar{x}) > 0 \) for some \( \bar{x} \) but \( Q(\bar{x}) < 0 \) for some other \( \bar{x} \).

a) In the special case \( n = 2 \) give (simple!) examples of matrices \( A \) that are positive definite, negative definite, and indefinite.

**Solution:** Several examples. Begin with the polynomial, not the matrix.

**Positive definite:** If \( \langle \bar{x}, A\bar{x} \rangle = x_1^2 + x_2^2 \) then \( A \) is the identity matrix \( I \), and \( \langle \bar{x}, A\bar{x} \rangle = 2x_1^2 + 3x_2^2 \) so \( A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \).

**Negative definite:** For \( \langle \bar{x}, A\bar{x} \rangle = -x_1^2 - x_2^2 \), the matrix is \( -I \) while for \( \langle \bar{x}, A\bar{x} \rangle = -2x_1^2 - 3x_2^2 \), the matrix is \( \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \).

**Indefinite:** For \( \langle \bar{x}, A\bar{x} \rangle = x_1^2 - x_2^2 \) the matrix is \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) while for \( \langle \bar{x}, A\bar{x} \rangle = -2x_1^2 + 5x_2^2 \) the matrix is \( \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \).

**Note:** If \( \langle \bar{x}, A\bar{x} \rangle = 3x_2^2 \), the matrix is \( A := \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \) is not positive definite, it is **positive semi-definite**, that is, \( \langle \bar{x}, A\bar{x} \rangle \geq 0 \) for all \( \bar{x} \) but \( \langle \bar{x}, A\bar{x} \rangle = 0 \) for some \( \bar{x} \neq 0 \).

b) In the special case where \( A \) is an invertible **diagonal** matrix,

\[
A = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix},
\]

under what conditions is \( Q(\bar{x}) \) positive definite, negative definite, and indefinite?

**Remark:** We will see that the general case can always be reduced to this special case where \( A \) is diagonal.

**Solution:** Key step: here

\[
\langle \bar{x}, A\bar{x} \rangle = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.
\]
If we let \( \vec{x} = (0, 1, 0, \ldots, 0) \), clearly \( \langle \vec{x}, A\vec{x} \rangle = \lambda_2 \) so if \( A \) is positive definite, then \( \lambda_2 > 0 \). Similarly, if \( A \) is positive definite, then all the \( \lambda_j \) are positive.

Conversely, if all the \( \lambda_j \) are positive, it is clear that \( A \) is positive definite. By the same reasoning, \( A \) is negative definite if (and only if) all the \( \lambda_j < 0 \), and indefinite if at least one \( \lambda_j \) is positive and another is negative.

**Note:** the assumption “\( A \) is invertible” implies that none of the \( \lambda_j \) are zero.

**Bonus Problems**

[Please give this directly to Professor Kazdan]

B-1 Let \( S := \{u(x) \in C^2[0, \pi] \text{ with } u(0) = u(\pi) = 0\} \) and let \( Lu := -u''(x) \). Use the inner product \( \langle u, v \rangle = \int_0^\pi u(x)v(x)dx \).

a) If \( u \) and \( v \) are in \( S \), show that \( \langle Lu, v \rangle = \langle u, Lv \rangle \). This shows that \( L \) is self-adjoint on this space of functions. [**Hint:** Integrate by parts.]

**Solution:** Using integration by parts you obtain \( \langle Lu, v \rangle = \int_0^\pi u''v' \, dx \) and \( \langle v, Lu \rangle = \langle Lu, v \rangle = \int_0^\pi u'v' \, dx \).

b) If \( u(x) \in S \), \( u \not\equiv 0 \), is an eigenfunction of \( L \), so \( Lu = \lambda u \) for some constant \( \lambda \), show that \( \lambda > 0 \). [**Hint:** Compute \( \langle Lu, u \rangle \) and integrate by parts.]

**Solution:** If \( \lambda = 0 \) then \( u \) solves \( u'' = 0 \) and get \( u \notin S \) so we have a contradiction. Hence \( \lambda \neq 0 \). Now, \( \lambda \langle u, u \rangle = \langle Lu, u \rangle = \int_0^\pi (u')^2 \, dx \geq 0 \) Hence \( \lambda \geq 0 \). Thus \( \lambda > 0 \).

c) Find the eigenvalues \( \lambda_k \) and eigenfunctions \( u_k(x) \) of \( L \) (remember to use the boundary conditions \( u(0) = u(\pi) = 0 \)).

**Solution:** For this part see to the notes:

[http://hans.math.upenn.edu/~kazdan/312S13/notes/Lu=-DDu.pdf](http://hans.math.upenn.edu/~kazdan/312S13/notes/Lu=-DDu.pdf)

B-2 Let \( A : \mathbb{R}^n \to \mathbb{R}^k \) be a linear map that is onto but not one-to-one. Say \( X_1 \) is a solution of \( AX = Y \). Is there a “best” possible solution? What can one say? Think about this before reading the next paragraph.

a) Show that \( AA^* \) is invertible so there is exactly one solution \( V \) of \( AA^*V = Y \). Thus the vector \( X_2 := A^*V \) is also a solution of \( AX = Y \).

**Solution:** Since \( A \) is onto we have that \( A^* \) is one-to-one, namely \( \ker A^* = \{0\} \) and hence that the square matrix \( AA^* \) is invertible. [This is the same as Problem 11d) above.]

b) Show that if \( X_1 \) is any solution of \( AX = Y \), then \( X_2 \) is closer to the origin, that is, \( \|X_2\| \leq \|X_1\| \). In other words, \( X_2 \) is the solution that is closest to the origin. [**Hint:** the general solution of \( AX = Y \) is \( X = X_2 + Z \) where \( Z \in \ker A \).]
Solution: We have that $X_2 = A^*V \in \text{im} A^* = (\ker A)^\perp$ and $X_1 = X_2 + Z$ for some $Z \in \ker A$, hence $Z$ and $X_2$ are orthogonal. Then by the Pythagorean theorem we have that

$$\|X_1\|^2 = \|X_2 + Z\|^2 = \|X_2\|^2 + \|Z\|^2 \geq \|X_2\|^2.$$