Problem Set 7

DUE: In class Thurs, March 20. Late papers will be accepted until 1:00 PM Friday.

For the week after Spring Break, please read Chapter 6 on determinants and, as a review of Math 240, Chapter 7.1 and 7.2.

1. For real \( c > 0, \ c \neq 1 \), and distinct points \( \vec{p} \) and \( \vec{q} \) in \( \mathbb{R}^n \), consider the points \( \vec{x} \in \mathbb{R}^n \) that satisfy
\[
\| \vec{x} - \vec{p} \| = c \| \vec{x} - \vec{q} \| .
\]
Show that these points lie on some sphere: \( \| \vec{x} - \vec{x}_0 \| = r \), with center at \( \vec{x}_0 \) and radius \( r \). This problem is to find the center and radius of the sphere in terms of \( \vec{p}, \vec{q}, \) and \( c \).
What happens in the special case \( c = 1 \)?

2. [Bretscher, Sec. 5.5 #24]. Using the inner product \( \langle f, g \rangle := \int_0^1 f(x)g(x)\,dx \), for certain polynomials \( f, g, \) and \( h \) say we are given the following table of inner products:

<table>
<thead>
<tr>
<th>\langle \cdot, \cdot \rangle</th>
<th>f</th>
<th>g</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>g</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>h</td>
<td>8</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

For example, \( \langle g, h \rangle = \langle h, g \rangle = 3 \). Let \( E \) be the span of \( f \) and \( g \).

a) Compute \( \langle f, g + h \rangle \).

b) Compute \( \| g + h \| \).

c) Find proj\( E \)h. [Express your solution as linear combinations of \( f \) and \( g \).]

d) Find an orthonormal basis of the span of \( f, g, \) and \( h \) [Express your results as linear combinations of \( f, g, \) and \( h \)].

3. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Find the conditions on the coefficients \( a, b, c, \) and \( d \) so that \( A \) is an orthogonal matrix.

4. [See Homework 4, Problem 13] Say \( X(t) \) is a solution of the differential equation \( \frac{dX}{dt} = AX \), where \( A \) is an anti-symmetric matrix. Show that \( \| X(t) \| = \text{constant} \). [Remark: A special case is that \( X(t) := \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \) satisfies \( X' = AX \) with \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) so this problem gives another proof that \( \cos^2 t + \sin^2 t = 1 \).]
5. [Bretscher, Sec. 5.4 #20] Using pencil and paper, find the least-squares solution to \( A\vec{x} = \vec{b} \) where
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.
\]

6. Use the Method of Least Squares to find the parabola \( y = ax^2 + b \) that best fits the following data given by the following four points \((x_j, y_j), j = 1, \ldots, 4:\)
\[
(-2, 4), \quad (-1, 3), \quad (0, 1), \quad (2, 0).
\]
Ideally, you’d like to pick the coefficients \( a \) and \( b \) so that the four equations \( ax_j^2 + b = y_j, \)
\( j = 1, \ldots, 4 \) are all satisfied. Since this probably can’t be done, one uses least squares to find the best possible \( a \) and \( b. \)

7. The water level in the North Sea is mainly determined by the so-called M2 tide, whose period is about 12 hours see [https://en.wikipedia.org/wiki/Tide]. The height \( H(t) \) thus roughly has the form
\[
H(t) = c + a \sin(2\pi t/12) + b \cos(2\pi t/12),
\]
where time \( t \) is measured in hours (note \( \sin(2\pi t/12) \) and \( \cos(2\pi t/12) \) are periodic with period 12 hours). Say one has the following measurements:

<table>
<thead>
<tr>
<th>( t ) (hours)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(t) ) (meters)</td>
<td>1.0</td>
<td>1.6</td>
<td>1.4</td>
<td>0.6</td>
<td>0.2</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Use the method of least squares to find the constants \( a, b, \) and \( c \) in \( H(t) \) for this data. [Although one can certainly solve this “by hand,” you may prefer to use computer software.]

8. Some experimental data \((x_i, y_i)\) is believed to fit a curve of the form
\[
y = \frac{1 + x}{a + bx^2},
\]
where the parameters \( a \) and \( b \) are to be determined from the data. The method of least squares does not apply directly to this since the parameters \( a \) and \( b \) do not appear linearly. Show how to find a modified equation to which the method of least squares does apply.

9. **Plotting graphs** This problem concerns the straight line in the plane that passes through the two points \((4, 0)\) and \((0, 2)\) (draw a sketch).
a) If the horizontal axis is $x$ and the vertical axis $y$, what is the equation for $y$ as a function of $x$?

b) If the horizontal axis is $\log x$ and the vertical axis $y$, what is the equation for $y$ as a function of $x$?

c) If the horizontal axis is $x$ and the vertical axis $\log y$, what is the equation for $y$ as a function of $x$?

d) If the horizontal axis is $\log x$ and the vertical axis $\log y$, what is the equation for $y$ as a function of $x$?

10. Say $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with the property that $A^2 - 3A + 2I = 0$. If $\vec{v} \neq 0$ is an eigenvector of $A$ with eigenvalue $\lambda$, that is, $A\vec{v} = \lambda\vec{v}$, what are the possible values of $\lambda$?

11. Let $R$ be a (real) $3 \times 3$ orthogonal matrix.

   a) Show that the eigenvalues, $\lambda$, which may be complex, all have absolute value 1.

   b) If $\det R = 1$ show that $\lambda = 1$ is one of the eigenvalues of $R$ and that if $R \neq I$, no other eigenvalue can be 1.

   For the remainder of this problem assume $\det R = 1$ and $R \neq I$.

   c) Let $N$ be an eigenvector corresponding to $\lambda = 1$ and let $Q$ be the plane of all vectors orthogonal to $N$. Show that $R$ maps $Q$ to $Q$.

   d) Why does this show that $R$ is a rotation of the plane $Q$ with $N$ as the axis of rotation?

12. Let $A$ be a $3 \times 3$ matrix with eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ and corresponding linearly independent eigenvectors $V_1$, $V_2$, $V_3$ which we can therefore use as a basis.

   a) If $X = av_1 + bv_2 + cv_3$, compute $AX$, $A^2X$, and $A^{35}X$ in terms of $\lambda_1$, $\lambda_2$, $\lambda_3$, $V_1$, $V_2$, $V_3$, $a$, $b$ and $c$ (only).

   b) If $\lambda_1 = 1$, $|\lambda_2| < 1$, and $|\lambda_3| < 1$, compute $\lim_{k \to \infty} A^kX$. Explain your reasoning clearly.
**Bonus Problem**
[Please give this directly to Professor Kazdan]

**B-1** Let \( P_1, P_2, \ldots, P_k \) be \( k \) points (think of them as *data*) in \( \mathbb{R}^3 \) and let \( S \) be the plane

\[
S := \{ X \in \mathbb{R}^3 : \langle X, N \rangle = c \},
\]

where \( N \neq 0 \) is a unit vector normal to the plane and \( c \) is a real constant. [Note: This plane \( S \) is only a subspace if \( c = 0 \). The subspace parallel to this plane satisfies \( \langle X, N \rangle = 0 \) so, writing more carefully, \( N \) is orthogonal to \( S \) only when \( c = 0 \). Indeed, a preliminary first step (see below), is to make a change of coordinates to reduce to this special case.]

This problem outlines how to find the plane that *best approximates the data points* in the sense that it minimizes the function

\[
Q(N, c) := \sum_{j=1}^{k} \text{distance}(P_j, S)^2.
\]

Determining this plane means finding \( N \) and \( c \).

a) Show that for a given point \( P \), then

\[
\text{distance}(P, S) = |\langle P - X, N \rangle| = |\langle P, N \rangle - c|,
\]

where \( X \) is any point in \( S \)

b) First do the special case where the center of mass \( \bar{P} := \frac{1}{k} \sum_{j=1}^{k} P_j \) is at the origin, so \( \bar{P} = 0 \). Show that for any \( P \), then \( \langle P, N \rangle^2 = \langle N, PP^*N \rangle \). Here view \( P \) as a column vector so \( PP^* \) is a \( 3 \times 3 \) matrix.

Use this to observe that the desired plane \( S \) is determined by letting \( N \) be an eigenvector of the matrix

\[
A := \sum_{j=1}^{k} P_j P_j^T
\]

corresponding to its lowest eigenvalue. What is \( c \) in this case?

c) Reduce the general case to the previous case by letting \( V_j = P_j - \bar{P} \).

d) Find the equation of the line \( ax + by = c \) that, in the above sense, best fits the data points \((-1, 3), (0, 1), (1, -1), (2, -3)\).

e) Let \( P_j := (p_{j1}, \ldots, p_{j3}), j = 1, \ldots, k \) be the coordinates of the \( j \text{th} \) data point and \( Z_\ell := (p_{1\ell}, \ldots, p_{k\ell}), \ell = 1, \ldots, 3 \) be the vector of \( \ell \text{th} \) coordinates. If \( a_{ij} \) is the \( ij \) element of \( A \), show that \( a_{ij} = \langle Z_i, Z_j \rangle \). Note that this exhibits \( A \) as a *Gram matrix*.

[Last revised: March 18, 2014]