DIRECTIONS: This exam has three parts. Part A has 4 True-False questions, Part B has 3 short answer questions, and Part C has 6 traditional problems. Each problem is worth 10 points. 130 points total.

To receive full credit your solution must be clear and correct. No fuzzy reasoning. Partial credit will only be given for the problems in Part C. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 × 5 card with notes. Please box your answers where appropriate.

Note: To be fair to everyone, those who submit their exam paper late (after 1:20) will be “charged” 5 points for every 2 additional minutes.

PART A. Four True-False questions. 10 points each, so 2 points for each item.

A-1. Say you have \( k \) linear algebraic equations in \( n \) variables; in matrix form we write \( AX = Y \).

T  F  If \( n = k \) there is always \textit{at most one} solution.
T  F  If \( n > k \) you can \textit{always} solve \( AX = Y \).
T  F  If \( n > k \) the nullspace of \( A \) has dimension greater than zero.
T  F  If \( n < k \) then for \textit{some} \( Y \) there is no solution of \( AX = Y \).
T  F  If \( n < k \) the \textit{only} solution of \( AX = 0 \) is \( X = 0 \).

ANSWERS:  F  F  T  T  F

A-2. Given two \( n \times n \) matrices \( A \) and \( B \) with \( AB = 0 \), which of the following assertions \textit{must} be true?

T  F  \( BA = 0 \)
T  F  Either \( A = 0 \) or \( B = 0 \) (or both).
T  F  If \( \det A = -3 \), then \( B = 0 \).
T  F  If \( B \) is invertible then \( A = 0 \).
T  F  There is a vector \( V \neq 0 \) such that \( BAV = 0 \).

ANSWERS:  F  F  T  T  T

In greater detail: For Parts 1 and 2 consider \( A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), and \( B := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Then \( AB = 0 \) but \( BA = A \neq 0 \).

For Part 3, we have \( \det A \neq 0 \) so \( A \) is invertible. Thus \( AB = 0 \) implies that \( 0 = A^{-1}(AB) = B \). Part 4 is similar: \( 0 = (AB)B^{-1} = A \).

For Part 5, \( \det(BA) = \det(AB) = 0 \) so \( BA \) is not invertible; hence its nullspace must have a non-zero vector \( V \).

Score

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A-3. Circle T for each of the following sets that are linear spaces.

T F \{X = (x_1, x_2, x_3) in R^3 with the property x_1 - 2x_3 = 0\}
T F The set of solutions x to the system Ax = 0, where A is an m x n matrix.
T F The set of 2 x 2 matrices A with det(A) = 0.
T F The set of polynomials p(x) with \int_1^1 p(x) dx = 0.
T F The set of solutions y = y(t) of the differential equation y'' + y' + y = 0.

ANSWERS: T T F T T

A-4. Circle T for each of the following sets of vectors that are bases for R^2.

T F \{(0, 1), (1, 1)\}
T F \{(1, 0), (0, 1), (1, 1)\}
T F \{(1, 0), (-1, 0)\}
T F \{(1, 1), (1, -1)\}
T F \{((1, 1), (2, 2)\}

ANSWERS: T F F T F

Part B. Three short-answer questions. 10 points each. Partial credit will rarely be given. Please box your answers where appropriate.

B-1. Let A be an invertible square matrix. Say there is a vector V with the property that AV = 7V. Compute A^2V and A^{-1}V.

ANSWERS: A^2V = 49V, A^{-1}V = \frac{1}{7}V

B-2. Let A be an n x n matrix. Which of the following statements are equivalent to “the matrix A is invertible”?

(a) The columns of A are linearly independent.
(b) The linear transformation T_A : R^n → R^n defined by A is 1-1.
(c) The rank of A is n.

(A) a and b only (B) b and c only (C) a and c only (D) a, b and c
(E) a only (F) None

ANSWER: (D)

B-3. Let A be a matrix, not necessarily square. Say V and W are particular solutions of the equations AV = Y_1 and AW = Y_2, respectively, while Z ≠ 0 is a solution of the homogeneous equation AZ = 0. Answer the following in terms of V, W, and Z.

a) Find some solution of AX = 3Y_1 - 5Y_2.

ANSWER: X = 3V - 5W
b) Find another solution (other than \(Z\) and 0) of the homogeneous equation \(AX = 0\).

**ANSWER:** \(X = \frac{1}{2}Z\), in fact, \(X = cZ\) for any scalar \(c\).

c) Find two solutions of \(AX = \mathbf{Y}_1\).

**ANSWER:** \(X = V\) and \(X = V + Z\). Better: \(X = V + cZ\) for any scalar \(c\).

d) Find another solution of \(AX = 3\mathbf{Y}_1 - 5\mathbf{Y}_2\).

**ANSWER:** \(X = 3V - 5W + cZ\) for any scalar \(c\).

e) If \(A\) is a square matrix, then \(\det A = ?\)

**ANSWER:** Since the homogeneous equation has a non-trivial solution \(Z\), then \(\det A = 0\).

**PART C.** Six problems. 10 points each. Please box your answers where appropriate.

C-1. A linear transformation \(T : \mathbb{R}^3 \to \mathbb{R}^3\) first rotates the \(xy\)-plane by \(+90^\circ\) (leaving the \(z\)-axis fixed), followed by an orthogonal projection onto the \(yz\)-plane. Find the standard matrix representation for \(T\).

**ANSWERS:** The columns of \(T\) are the images of the standard basis vectors. But Under this map

\[
\begin{align*}
(1,0,0) &\to (0,1,0) \\
(0,1,0) &\to (-1,0,0) \\
(0,0,1) &\to (0,0,1)
\end{align*}
\]

Thus these are the columns of \(T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\).

C-2. Consider the system of equations

\[
\begin{align*}
x + y - z &= a \\
x - y + 2z &= b \\
3x + y &= c
\end{align*}
\]

a) Find the general solution of the homogeneous equation.

**ANSWER:** We solve the equations when \(a = b = c = 0\). The result is \(x = -\frac{1}{2}z\) and \(y = \frac{3}{2}z\). Thus the general solution is \((x, y, z) = (-1, 3, 2)c\) for any constant \(c\).

b) Observe that \(x = 1, y = 1, z = 1\) is a particular solution of the inhomogeneous equations when \(a = 1, b = 2, \text{ and } c = 4\). Find the most general solution of these inhomogeneous equations.

**ANSWER:** Since the general solution \(X\) of the inhomogeneous equation is the sum of a particular solution of the inhomogeneous equation plus the general solution of the homogeneous equation, then

\[X = (1,1,1) + (-1,3,2)c\] for any constant \(c\).

*If you answered this part by again solving the equations directly, you have not understood a significant part of the course.*
c) If \( a = 1, \ b = 2, \) and \( c = 3, \) show these equations have no solution.

**ANSWER:** For example from echelon form (or just adding twice the first equation to the second equation) one sees that if a solution exists, then \( 2a + b = c. \) This is not satisfied for \( a = 1, \ b = 2, \ c = 3. \)

d) If \( a = 0, \ b = 0, \ c = 1, \) show the equations have no solution. [Note: \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} \).]

**ANSWER:** Method 1. Think of these equations as defining a linear map from \((x, yz)\) to \((a, b, c. \)

We know that the image of a linear map is a linear space. We also know that \((1, 2, 4)\) is in the image. If \((0, 0, 1)\) were in the image, then so would \((1, 2, 4) - (0, 0, 1) = (1, 2, 3). \) But in part c) we showed that \((1, 2, 3)\) is not in the image. Thus \((0, 0, 1)\) can’t be in the image either.

**Method 2.** The approach in part c) also works here.

e) Let \( A \) be the matrix associated with these equations, \( A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}. \) Compute \( \det A \)

using any method (one method involves no computation at all).

**ANSWER:** From part a) the nullspace of \( A \) is not zero. Thus \( A \) is not invertible so \( \det A = 0. \)

[Of course one could also compute this the long way. While that is not too painful here, had the matrix \( A \) been much larger, this would have been very time-consuming.]

**C-3.** Let \( C \) and \( B \) be square matrices with \( C \) invertible.

\[ (C B C^{-1})^2 = C (B^2) C^{-1} \]

**ANSWER:** \( (C B C^{-1})^2 = (C B C^{-1})(C B C^{-1}) = C B (C^{-1} C) B C^{-1} = C B B C^{-1} = C B^2 C^{-1}. \)

\[ (C B^{-2} C^{-1}) \]

**ANSWER:** Let \( D = C B^{-2} C^{-1}. \) To show that \( D \) is the inverse of \( C B^2 C^{-1} \) we show that \( D (C B^2 C^{-1}) = I. \)

But

\[ D (C B^2 C^{-1}) = (C B^{-2} C^{-1})(C B^2 C^{-1}) = C B^{-2} (C^{-1} C) B^2 C^{-1} = C B^{-2} B^2 C^{-1} = C C^{-1} = I. \]

A slightly different approach. We use that for an invertible matrix \( C^{-2} = (C^{-1})^2. \) Thus

\[ (C B C^{-1})^{-2} = ((C B C^{-1})^{-1})^2 = (C B^{-1} C^{-1})^2 = (C B^{-1} C^{-1})(C B^{-1} C^{-1}) = C B^{-2} C^{-1}. \]

**C-4.** Let \( \lambda \) be a scalar parameter and consider the system \[
\begin{align*}
x + \lambda y &= 0 \\ \lambda x + 4y &= 0
\end{align*}
\]

(a) Use row reduction to determine the values of \( \lambda \) for which the system has infinitely many solutions.

**ANSWER:** Multiplying the first equation by \( \lambda \) and subtracting this from the second equation we find that \((4 - \lambda^2) y = 0. \) Thus either \( y = 0 \) or \( \lambda^2 = 4. \) But if \( y = 0, \) then \( x = 0 \) so there is only one solution. Thus we must have \( \lambda^2 = 4, \) that is \( \lambda = \pm 2. \)

(b) For each of the values of \( \lambda \) found in (a), sketch the set of solutions in the \( xy \)-plane.
ANSWER: If $\lambda = 2$ then $x = -2y$, while if $\lambda = -2$ then $x = 2y$. These straight lines (through the origin) define two one-dimensional subspaces.

(c) Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. For which values of $\lambda$ is the nullspace of $A$ not just the zero vector?

ANSWER: This is just a restatement of part a)(Why?), so we have $\lambda = \pm 2$.

C-5. Let the invertible square matrix $A$ have the property that its inverse equals its transpose. Show that $\det A = \pm 1$.

ANSWER: First we recall that $\det A^T = \det A$ and that $\det A^{-1} = 1/\det A$.

We are given that $A^{-1} = A^T$. Taking the determinant of both sides and using the previous sentence we obtain

$$\frac{1}{\det A} = \det A.$$

Thus $1 = (\det A)^2$ so $\det A = \pm 1$.

Slightly different method: Multiply $A^{-1} = A^T$ by $A$ to find $I = AA^T$. Then taking determinants $1 = \det(AA^T) = (\det A)(\det A^T) = (\det A)^2$ so $\det A = \pm 1$.

C-6. Answer each of the following questions with justification; that is, give a proof if the statement is true or provide an example to show that it is not. In each case your answers should be brief.

(a) Suppose that $u, v$ and $w$ are vectors in a vector space $V$ and $T: V \to W$ is a linear map. If $u, v$ and $w$ are linearly dependent, is it true that $T(u), T(v), T(w)$ are linearly dependent?

ANSWER: No. Let $T$ be the zero map. In fact, This is true only for maps $T$ that are one-to-one since if $Z \neq 0$ is in the nullspace of $T$, then the set consisting of the single vector $\{Z\}$ is linearly independent while the set consisting of $\{TZ\}$ is trivially linearly dependent since it is the zero vector.

(b) If $T: \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map is it possible that the nullspace of $T$ is one dimensional?

ANSWER: No, since by the Rank Theorem, $\dim N(T) + \dim R(T) = 6$ and $\dim R(T) \leq 4$, we have $\dim N(T) \geq 6 - 4 = 2$.

**Bonus Problem.** Let $A: \mathbb{R}^3 \to \mathbb{R}^2$ and $B: \mathbb{R}^2 \to \mathbb{R}^3$, so $BA: \mathbb{R}^3 \to \mathbb{R}^3$. Show that $BA$ can not be invertible.

ANSWER: Note that since neither $A$ nor $B$ are square matrices, if you mention $A^{-1}, B^{-1}, \det A,$ or $\det B$, you have already made a fatal error.

We give two of the many ways to solve this problem. Write $C := BA$.

**Method 1.** We show that the nullspace of $C$ is not zero – and hence that it can’t be invertible. Since the homogeneous system $Az = 0$ has 2 equations in 3 unknowns, then it has a non-trivial solution. But then $Cz = BAz = B0 = 0$, so the nullspace of $C$ is not zero.

**Method 2.** We show that the $3 \times 3$ matrix $BA$ has rank $BA \leq 2$, and hence that $BA$ is not onto, so it is not invertible. But the rank $B \leq 2$ and range $BA \subset$ range $B$ (why?) so rank $BA \leq 2$. 