DIRECTIONS: This exam has three parts. Part A has 4 True-False questions, Part B has 3 short answer questions, and Part C has 6 traditional problems. Each problem is worth 10 points. 130 points total.

To receive full credit your solution must be clear and correct. No fuzzy reasoning. Partial credit will only be given for the problems in Part C. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 × 5 card with notes. Please box your answers where appropriate.

NOTE: To be fair to everyone, those who submit their exam paper late (after 1:20) will be “charged” 5 points for every 2 additional minutes.

PART A. Four True-False questions. 10 points each, so 2 points for each item.

A-1. Say you have \( k \) linear algebraic equations in \( n \) variables; in matrix form we write \( AX = Y \).

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<th>T</th>
<th>F</th>
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<tr>
<td>If ( n = k ) there is always at most one solution.</td>
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<td>If ( n &gt; k ) you can always solve ( AX = Y ).</td>
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<td>If ( n &gt; k ) the nullspace of ( A ) has dimension greater than zero.</td>
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<td>If ( n &lt; k ) then for some ( Y ) there is no solution of ( AX = Y ).</td>
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<td>If ( n &lt; k ) the only solution of ( AX = 0 ) is ( X = 0 ).</td>
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<td>C-6</td>
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Total
A-3. Circle T for each of the following sets that are linear spaces.

\begin{itemize}
  \item [T] \quad F \quad \{X = (x_1, x_2, x_3) \text{ in } R^3 \text{ with the property } x_1 - 2x_3 = 0\}
  \item [T] \quad F \quad \text{The set of solutions } x \text{ to the system } Ax = 0, \text{ where } A \text{ is an } m \times n \text{ matrix.}
  \item [T] \quad F \quad \text{The set of } 2 \times 2 \text{ matrices } A \text{ with } \det(A) = 0.
  \item [T] \quad F \quad \text{The set of polynomials } p(x) \text{ with } \int_{-1}^{1} p(x) dx = 0.
  \item [T] \quad F \quad \text{The set of solutions } y = y(t) \text{ of the differential equation } y'' + y' + y = 0.
\end{itemize}

A-4. Circle T for each of the following sets of vectors that are bases for \( R^2 \).

\begin{itemize}
  \item [T] \quad F \quad \{(0, 1), (1, 1)\}
  \item [T] \quad F \quad \{(1, 0), (0, 1), (1, 1)\}
  \item [T] \quad F \quad \{(1, 0), (-1, 0)\}
  \item [T] \quad F \quad \{(1, 1), (1, -1)\}
  \item [T] \quad F \quad \{((1, 1), (2, 2))\}
\end{itemize}

**PART B.** Three short-answer questions. 10 points each. Partial credit will rarely be given. Please \boxed{\text{box}} your answers where appropriate.

B-1. Let \( A \) be an invertible square matrix. Say there is a vector \( V \) with the property that \( AV = 7V \). Compute \( A^2V \) and \( A^{-1}V \).
B-2. Let $A$ be an $n \times n$ matrix. Which of the following statements are equivalent to “the matrix $A$ is invertible”?

(a) The columns of $A$ are linearly independent.

(b) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $A$ is 1-1.

(c) The rank of $A$ is $n$.

(A) a and b only  (B) b and c only  (C) a and c only  (D) a, b and c
(E) a only  (F) None

B-3. Let $A$ be a matrix, not necessarily square. Say $V$ and $W$ are particular solutions of the equations $AV = Y_1$ and $AW = Y_2$, respectively, while $Z \neq 0$ is a solution of the homogeneous equation $AZ = 0$. Answer the following in terms of $V$, $W$, and $Z$.

a) Find some solution of $AX = 3Y_1 - 5Y_2$.

b) Find another solution (other than $Z$ and 0) of the homogeneous equation $AX = 0$.

c) Find two solutions of $AX = Y_1$.

d) Find another solution of $AX = 3Y_1 - 5Y_2$.

e) If $A$ is a square matrix, then $\det A =$?

PART C. Six problems. 10 points each, Please [box] your answers where appropriate.

C-1. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ first rotates the $xy$-plane by $+90^\circ$ (leaving the $z$-axis fixed), followed by an orthogonal projection onto the $yz$-plane. Find the standard matrix representation for $T$. 
C-2. Consider the system of equations

\[
\begin{align*}
    x + y - z &= a \\
    x - y + 2z &= b \\
    3x + y &= c
\end{align*}
\]

a) Find the general solution of the homogeneous equation.

b) Observe that \( x = 1, \ y = 1, \ z = 1 \) is a particular solution of the inhomogeneous equations when \( a = 1, \ b = 2, \) and \( c = 4. \) Find the most general solution of these inhomogeneous equations.

c) If \( a = 1, \ b = 2, \) and \( c = 3, \) show these equations have no solution.

d) If \( a = 0, \ b = 0, \) and \( c = 1, \) show the equations have no solution. [Note: \( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}) - (\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}) \).]

e) Let \( A \) be the matrix associated with these equations, \( A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}. \) Compute \( \det A \) using any method (one method involves no computation at all).
C-3. Let $C$ and $B$ be square matrices with $C$ invertible.

a) Show that $(CBC^{-1})^2 = C(B^2)C^{-1}$

b) If $B$ is also invertible, is it true that $(CBC^{-1})^{-2} = C(B^{-2})C^{-1}$? Why?

C-4. Let $\lambda$ be a scalar parameter and consider the system

\[
\begin{aligned}
  x + \lambda y &= 0 \\
  \lambda x + 4y &= 0 
\end{aligned}
\]

(a) Use row reduction to determine the values of $\lambda$ for which the system has infinitely many solutions.

(b) For each of the values of $\lambda$ found in (a), sketch the set of solutions in the $xy$-plane.

(c) Let $A = \begin{pmatrix} 1 & \lambda \\ \lambda & 4 \end{pmatrix}$. For which values of $\lambda$ is the nullspace of $A$ not just the zero vector?
C-5. Let the invertible square matrix $A$ have the property that its inverse equals its transpose. Show that $\det A = \pm 1$.

C-6. Answer each of the following questions with justification; that is, give a proof if the statement is true or provide an example to show that it is not. In each case your answers should be brief.

(a) Suppose that $u$, $v$ and $w$ are vectors in a vector space $V$ and $T : V \rightarrow W$ is a linear map. If $u$, $v$ and $w$ are linearly dependent, is it true that $T(u)$, $T(v)$ and $T(w)$ are linearly dependent?

(b) If $T : \mathbb{R}^6 \rightarrow \mathbb{R}^4$ is a linear map is it possible that the nullspace of $T$ is one dimensional?

**Bonus Problem.** Let $A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $B : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, so $BA : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Show that $BA$ can not be invertible.