Homework Set 6, Due Thursday, Feb. 24, 2005

(Late papers will be accepted until 4 PM on Fri. Feb. 25)

1. Strang p. 180 #1, 2, 14

2. Strang p. 180 #4, 5

3. Strang p. 181 #10, 11, 17

4. Strang p. 182 #21, 22

5. Strang p. 191 #6

6. Strang p. 192 #10

7. A linear map \( R : V \to V \) acting on a vector space \( V \) is called a reflection if \( R^2 = I \).
   
   a) Show that the matrix \( R = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \) is a reflection. Draw a sketch of \( \mathbb{R}^2 \) showing the vectors \((1,2), (-1,0), (0,3)\) and their images under \( R \). Also indicate both the subspace \( U \) of vectors that are map to themselves: \( Ru = u \), and the subspace \( W \) of vectors that are mapped to their opposites: \( Rw = -w \). [From your sketch it is clear that this \( R \) is not an orthogonal reflection which is when the subspaces \( U \) and \( W \) are orthogonal.]
   
   b) Show that the subspaces \( U \) and \( W \) are orthogonal (so this is an orthogonal projection) if and only if \( R = R^* \).

8. [Relation between projections and reflections]. Let \( P \) be a projection into a sub-space \( U \) and let \( W \) be the nullspace of \( P \). Clearly, every vector \( X \) and be written as

\[ X = PX + (I-P)X, \]

that is, \( X = X_1 + X_2 \),

where \( X_1 = PX \) and \( X_2 = (I-P)X \).

a) Show that \( X_1 \) is in \( U \), that \( X_2 \) is in \( W \), and that \( P^2X = PX \) for every vector \( X \).

b) Let \( R \) be the related reflection across \( U \), so if \( X \) is in \( U \), then \( RX = X \), while if \( X \) is in \( W \), then \( RX = -X \). Show that every vector \( X \) can be written as

\[ X = X_3 + X_4, \]
where \( X_3 \in U \) and \( X_4 \in W \). [SUGGESTION: Observe that if we have \( X_3 \) and \( X_4 \), then you can write \( RX \) in terms of \( X_3 \) and \( X_4 \). Use this to solve for \( X_3 \) and \( X_4 \) in terms of \( X \) and \( RX \).]

c) Show that \( R \) and \( P \) are related by the simple formula \( R = 2P - I \). This makes obvious the relation between parts (a) and (b) above.

**Problems on Least Squares**

9. Strang, p. 215 #1, 4

10. Strang, p. 217 #17, 18

11. Strang, p. 229-230 #11, 18, 24

12. Find a plane of the form \( z = ax + by + c \) that best fits the following five points:
\[
(0,0,1.1), (1,1,2), (0,1,-0.1), (1,0,3), (0,-1,2.1).
\]

13. a) Some experimental data \((x_i, y_i)\) is believed to fit a curve of the form
\[
y = \frac{1 + x}{a + bx^2}
\]

where the parameters \( a \) and \( b \) are to be determined from the data. The method of least squares does not apply directly to this since the parameters \( a \) and \( b \) do not appear linearly. Show how to find an equivalent equation to which the method of least squares does apply.

b) Repeat part a) for the logistic curve \( y = \frac{L}{1 + e^{a - bx}} \). Here the constant \( L \) is assumed to be known. [If \( b > 0 \), then \( y \) converges to \( L \) as \( x \) increases. Thus the value of \( L \) can often be estimated simply by eye-ball ing a plot of the data for large \( x \).]

14. The comet Tentax, discovered only in 1968, moves within the solar system. The following are observations of its position \((r, \theta)\) in a polar coordinate system with center at the sun (here \( \theta \) is an angle measured in degrees, \( r \) in million km):

| \( r \) | 2.70 | 2.00 | 1.61 | 1.20 | 1.02 |
| \( \theta \) | 48 | 67 | 83 | 108 | 126 |
By Kepler’s first law the comet should move in a plane orbit whose shape is either an ellipse or hyperbola (this assumes the gravitational influence of the planets is neglected). Thus the polar coordinates \((r, \theta)\) satisfy

\[
r = \frac{p}{1 - e \cos \theta}
\]

where \(p\) and \(e\) are parameters describing the orbit. Use the data to estimate \(p\) and \(e\) by the method of least squares. [HINT: Make some (simple) preliminary manipulation so the parameters \(p\) and \(e\) appear linearly so one can then apply the method of least squares.]

**Bonus Problem** Given a unit vector \(\mathbf{n} = ai + bj + ck\) in \(\mathbb{R}^3\), find an explicit formula for a \(3 \times 3\) matrix \(M\) that rotates a vector \(\mathbf{v} = xi + yj + zk\) in \(\mathbb{R}^3\), about the “\(n\) axis” through a specified angle \(\theta\). Thus, your matrix \(M\) will involve \(a\), \(b\), \(c\), and \(\theta\).

[Last revised: February 18, 2005]