Some of these problems use the same matrices as Homework Set 8. That should save you some time.

1. Let \( u(t) = (u_1(t), u_2(t)) \). Solve the differential equation \( \frac{du}{dt} = Au \) with \( u(0) = (1, 2) \) where for \( A \) you use the following matrices:
   a). \( \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \)
   b). \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)
   c). \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)
   d). \( \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix} \).

2. Find the general solution of \( \frac{du}{dt} = Au \), where
   \[ A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 4 & -2 \end{pmatrix} \].

3. [Strang, p. 299 #10]. Get \( G_k \) be a sequence of numbers with the property
   \[ G_{k+2} = \frac{1}{2}(G_{k+1} + G_k), \quad \text{with} \quad G_0 = a \quad \text{and} \quad G_1 = b. \]
   a) Find an explicit formula for \( G_k \) by diagonalizing an appropriate matrix.
   b) Compute \( \lim_{k \to \infty} G_k \) in terms of \( a \) and \( b \). [You may find it useful to try the special case where \( G_0 = 1 \) and \( G_1 = 3 \).

4. [Strang, p. 301 #27]. Say \( A = SAS^{-1} \), where \( \Lambda \) is a diagonal matrix, and \( B \) is the block matrix \( B = \begin{pmatrix} A & 0 \\ 0 & 2A \end{pmatrix} \). Diagonalize \( B \).

5. In Homework 7 we worked with \( \Delta_n = \det M_n \) be the determinant of an \( n \times n \) matrix \( M_n \) with \( a \)'s along the main diagonal and \( b \)'s on the two “off diagonals” directly above and below the main diagonal (this is a simple example of a tridiagonal matrix). Thus
   \[ M_5 = \begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix}. \]
   You showed that \( \Delta_n = a\Delta_{n-1} - b^2\Delta_{n-2} \).
   The task now is to find an explicit formula for \( \Delta_n \).
6. Let $A$ be a real $2 \times 2$ matrix with the property that $A^3 = I$.
   a) If $\lambda$ is an eigenvalue of $A$, show that $\lambda^3 = 1$.
   b) What are all possible values of the trace and determinant of $A$?
   c) Use this to all possible real matrices $A$ satisfying $A^3 = I$.

7. If $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = I + A$, compute $A^2$, $A^3$, $e^A$, and $B^2$, $B^3$, $e^B$.

8. Let $B$ be a real antisymmetric matrix. Show that $M := e^B$ is an orthogonal matrix.

9. Let $M$ be a diagonalizable real $n \times n$ matrix with (possible complex) eigenvalues $\lambda_1$, $\lambda_2, \ldots, \lambda_n$. If the real parts of these eigenvalues are all negative, show that $e^{Mt} \to 0$ as $t \to \infty$.

10. Let $A$ be a real square matrix. If $\lambda$ is a real eigenvalue of $A$ with corresponding eigenvector $V$, and $\mu \neq \lambda$ is a real eigenvalue of $A^T$ with corresponding eigenvector $W$, show that $V$ and $W$ are orthogonal: $\langle V, W \rangle = 0$. 