Problem Set 5

DUE: Thurs. Oct. 11 in class. [Late papers accepted (without penalty) until 1:00 PM Friday.]

Please read Sections 6.1, 6.2, 1nd 6.3 of Burden-Faires.

Most of these problems should be a review of the basic linear algebra of Math 240, but emphasizing thinking of a system of linear equations as a linear mapping. They should be very short. In class on Tuesday we’ll discuss this a bit.

Problems

1. If $A$ is a $5 \times 5$ matrix with $\det A = -1$, compute $\det(-2A)$.

2. Consider the system of equations

$$
\begin{align*}
  x + y - z &= a \\
  x - y + 2z &= b.
\end{align*}
$$

a) Find the general solution of the homogeneous equation, so $a = b = 0$.

b) A particular solution of the inhomogeneous equations when $a = 1$ and $b = 2$ is $x = 1, y = 1, z = 1$. Find the most general solution of the inhomogeneous equations.

c) Find some particular solution of the inhomogeneous equations when $a = -1$ and $b = -2$.

d) Find some particular solution of the inhomogeneous equations when $a = 3$ and $b = 6$.

[Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

3. Solve the given system – or show that no solution exists:

$$
\begin{align*}
  x + 2y &= 1 \\
  3x + 2y + 4z &= 7 \\
  -2x + y - 2z &= -1
\end{align*}
$$

4. Say you have $k$ linear algebraic equations in $n$ variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

a) If $n = k$ there is always at most one solution.

b) If $n > k$ you can always solve $AX = Y$. 

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c) If $n > k$ the homogeneous equation $AX = 0$ has at least one solution $X \neq 0$.

d) If $n < k$ then for some $Y$ there is no solution of $AX = Y$.

e) If $n < k$ the only solution of $AX = 0$ is $X = 0$.

5. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a real matrix, not necessarily square. If two rows of $A$ are the same, show that $A$ is not onto by finding a vector $y = (y_1, \ldots, y_k)$ that is not in the image of $A$. [Hint: This is a mental computation if you write out the equations $Ax = y$ explicitly.]

6. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a real matrix, not necessarily square. If two columns of $A$ are the same, show that $A$ is not one-to-one by finding a vector $x = (x_1, \ldots, x_n)$ that satisfies $Ax = 0$.

NOT Assigned – but might help with the others. The following $2 \times 2$ matrices are valuable examples that may be surprising.

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = PR = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

Geometrically, $P$ is an orthogonal projection onto the $x_1$ axis, that is, if $X = (x_1, x_2) \in \mathbb{R}^2$ is a (column) vector in the plane, then $PX$ is its orthogonal projection onto the $x_1$ axis. Similarly, $R$ is a rotation by 90 degrees clockwise.

Compute (and interpret geometrically):

\[ P^2, \quad P^3, \quad R^2, \quad R^3, \quad R^4, \quad PR, \quad RP, \quad C^2, \quad CP, \quad PC. \]

7. Let $A$ and $B$ be $n \times n$ matrices with $AB = 0$. Give a proof or counterexample for each of the following.

a) Either $A = 0$ or $B = 0$ (or both).

b) $BA = 0$

c) If $\det A = -3$, then $B = 0$.

d) If $B$ is invertible then $A = 0$.

e) There is a vector $V \neq 0$ such that $BAV = 0$.

8. Let $A$ be a $4 \times 4$ matrix with determinant 7. Give a proof or counterexample for each of the following.

\footnote{The computer graphics examples in \url{https://www.math.upenn.edu/~kazdan/320F18/notes/Maple/F1.pdf} may also be illuminating.}
a) For some vector \( b \) the equation \( A\mathbf{x} = b \) has exactly one solution.

b) For some vector \( b \) the equation \( A\mathbf{x} = b \) has infinitely many solutions.

c) For some vector \( b \) the equation \( A\mathbf{x} = b \) has no solution.

d) For all vectors \( b \) the equation \( A\mathbf{x} = b \) has at least one solution.

9. a) Find a \( 2 \times 2 \) matrix that rotates the plane by +45 degrees (+45 degrees means 45 degrees counterclockwise).

b) Find a \( 2 \times 2 \) matrix that rotates the plane by +45 degrees followed by a reflection across the horizontal axis.

c) Find a \( 2 \times 2 \) matrix that reflects across the horizontal axis followed by a rotation the plane by +45 degrees.

d) Find a matrix that rotates the plane through +60 degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

10. Find a real \( 2 \times 2 \) matrix \( A \) (other than \( A = I \)) such that \( A^5 = I \).

11. Proof or counterexample. In these \( L \) is a linear map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), so its representation will be as a \( 2 \times 2 \) matrix.

   a) If \( L \) is invertible, then \( L^{-1} \) is also invertible.

   b) If \( LV = 5V \) for all vectors \( V \), then \( L^{-1}W = (1/5)W \) for all vectors \( W \).

   c) If \( L \) is a rotation of the plane by 45 degrees counterclockwise, then \( L^{-1} \) is a rotation by 45 degrees clockwise.

   d) If \( L \) is a rotation of the plane by 45 degrees counterclockwise, then \( L^{-1} \) is a rotation by 315 degrees counterclockwise.

   e) The zero map (\( 0V = 0 \) for all vectors \( V \)) is invertible.

   f) The identity map (\( IV = V \) for all vectors \( V \)) is invertible.

   g) If \( L \) is invertible, then \( L^{-1}0 = 0 \).

   h) If \( LV = 0 \) for some non-zero vector \( V \), then \( L \) is not invertible.

   i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: \( L = L^{-1} \).

12. Let \( A \) be a matrix, not necessarily square. Say \( \mathbf{V} \) and \( \mathbf{W} \) are particular solutions of the equations \( A\mathbf{V} = \mathbf{Y}_1 \) and \( A\mathbf{W} = \mathbf{Y}_2 \), respectively, while \( \mathbf{Z} \neq 0 \) is a solution of the homogeneous equation \( A\mathbf{Z} = 0 \). Answer the following in terms of \( \mathbf{V} \), \( \mathbf{W} \), and \( \mathbf{Z} \).

   a) Find some solution of \( A\mathbf{X} = 3\mathbf{Y}_1 \).

   b) Find some solution of \( A\mathbf{X} = -5\mathbf{Y}_2 \).
c) Find some solution of \( AX = 3Y_1 - 5Y_2 \).
d) Find another solution (other than \( Z \) and 0) of the homogeneous equation \( AX = 0 \).
e) Find two solutions of \( AX = Y_1 \).
f) Find another solution of \( AX = 3Y_1 - 5Y_2 \).
g) If \( A \) is a square matrix, then det \( A =? \)
h) If \( A \) is a square matrix, for any given vector \( W \) can one always find at least one solution of \( AX = W \)? Why?

13. Let \( R, M, \) and \( N \) be linear maps from the (two dimensional) plane to the plane given in terms of the standard \( i, j \) basis vectors by:

\[
 Ri = j, \quad Rj = -i \quad Mi = -i, \quad Mj = j \quad Nv = -v \text{ for all vectors } v
\]

a) Describe (pictures?) the actions of the maps \( R, R^2, R^{-1}, M, M^2, M^{-1} \) and \( N \).
b) Describe the actions of the maps \( RM, MR, RN, NR, MN, \) and \( NM \) [here we use the standard convention that the map \( RM \) means first use \( M \) then \( R \)]. Which pairs of these maps commute?
c) Which of the following identities are correct—and why?

\begin{align*}
1) \quad & R^2 = N & 2) \quad & N^2 = I & 3) \quad & R^4 = I & 4) \quad & R^5 = R \\
5) \quad & M^2 = I & 6) \quad & M^3 = M & 7) \quad & MNM = N & 8) \quad & NMN = R
\end{align*}
d) Find matrices representing each of the maps \( R, R^2, R^{-1}, M, \) and \( N \).

14. a). Find a linear map of the plane, \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that does the following transformation of the letter \( F \) (here the smaller \( F \) is transformed to the larger one):

![Diagram of transformation]

b). Find a linear map of the plane that inverts this map, that is, it maps the larger \( F \) to the smaller.
15. Linear maps $F(X) = AX$, where $A$ is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

$$F(X) = V + AX,$$

where $V$ is a vector. Note that $F(0) = V$.

Find the vector $V$ and the matrix $A$ that describe each of the following mappings [here the light blue $F$ is mapped to the dark red $F$].

(a).

(b).

(c).

(d).

[Last revised: October 8, 2018]