Taylor’s Theorem - Integral Remainder

**Theorem** Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has $k + 1$ continuous derivatives in some neighborhood $U$ of $x = a$. Then for any $x \in U$

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

$$+ \frac{f^{(k)}(a)}{k!}(x - a)^k + \int_a^x f^{(k+1)}(t)\frac{(x - t)^k}{k!}dt.$$

**Remark** In this version, the error term involves an integral. Because of this, we assume that $f^{k+1}$ is continuous, whereas previously we only assumed this derivative exists. However, we get the valuable bonus that this integral version of Taylor’s theorem does not involve the essentially unknown constant $c$. This is vital in some applications.

**Proof:** For clarity, fix $x = b$. By the Fundamental Theorem of Calculus,

$$f(b) = f(a) + \int_a^b f'(t)\,dt.$$

We integrate by parts – with an intelligent choice of a constant of integration:

$$\begin{align*}
  u &= f' \\
  dv &= dt \\
  du &= f''\,dt \\
  v &= t - b
\end{align*}$$
Then

\[ f(b) = f(a) + f'(t)(t - b)\bigg|_{t=a}^{t=b} - \int_a^b f''(t)(t - b) \, dt \]

\[ = f(a) + f'(a)(b - a) + \int_a^b f''(t)(b - t) \, dt. \]

Repeat this integration by parts:

\[ u = f'' \quad dv = (b - t)dt \]
\[ du = f''(t)dt \quad v = -(b - t)^2/2 \]

to find

\[ f(b) = f(a) + f'(a)(b - a) - f''(a)\frac{(b - a)^2}{2} + \int_a^b f'''(t)\frac{(b - t)^2}{2} \, dt \]

\[ = f(a) + f'(a)(b - a) + f''(a)\frac{(b - a)^2}{2} + \int_a^b f'''(t)\frac{(b - t)^2}{2} \, dt. \]

To find the general formula we claimed, just repeat the integrations by parts. As an exercise, it is instructive to carry out one more step to obtain the formula for \( k = 3 \).