Taylor’s Theorem - Integral Remainder

**Theorem** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function that has \( k + 1 \) continuous derivatives in some neighborhood \( U \) of \( x = a \). Then for any \( x \in U \)

\[
f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \int_a^x f^{(k+1)}(t) \frac{(x - t)^k}{k!} \, dt.
\]

**Remark** In this version, the error term involves an integral. Because of this, we assume that \( f^{k+1} \) is continuous, whereas previously we only assumed this derivative exists. However, we get the valuable bonus that this integral version of Taylor’s theorem does not involve the essentially unknown constant \( c \). This is vital in some applications.

**Proof:** For clarity, fix \( x = b \). By the Fundamental Theorem of Calculus,

\[
f(b) = f(a) + \int_a^b f'(t) \, dt.
\]

We integrate by parts – with an intelligent choice of a constant of integration:

\[
\begin{align*}
u &= f' \\
dv &= dt \\
u &= t - b
\end{align*}
\]

Then

\[
f(b) &= f(a) + f'(t)(t - b) \bigg|_{t=a}^{t=b} - \int_a^b f''(t)(t - b) \, dt \\
&= f(a) + f'(a)(b - a) + \int_a^b f''(t)(b - t) \, dt.
\]

Repeat this integration by parts:

\[
\begin{align*}
u &= f'' \\
dv &= (b - t)dt \\
u &= -(b - t)^2/2
\end{align*}
\]

To find

\[
f(b) &= f(a) + f'(a)(b - a) - f''(t) \frac{(b - t)^2}{2} \bigg|_{t=a}^{t=b} + \int_a^b f'''(t) \frac{(b - t)^2}{2} \, dt \\
&= f(a) + f'(a)(b - a) + f''(a) \frac{(b - a)^2}{2} + \int_a^b f'''(t) \frac{(b - t)^2}{2} \, dt.
\]

To find the general formula we claimed, just repeat the integrations by parts. As an exercise, it is instructive to carry out one more step to obtain the formula for \( k = 3 \).