Problem Set 0 [Rust Remover]

Due: Never – but you will be expected to know this material.

This week: Please review all of Chapters 1–5 in the Marsden text.

1. Let \( a_n > 1 \) be a sequence of real numbers. If \( a_n \to A \), show that \( A \geq 1 \).

2. Let \( a_n \neq 0 \) be a sequence of real numbers. If \( a_n \to A \) and \( A \neq 0 \), show that \( \frac{1}{a_n} \) converges to \( \frac{1}{A} \).

3. In \( \mathbb{R} \), if \( a_n \to A \) and \( b_n \to B \), show that the product \( a_nb_n \to AB \).

   If instead you assume that \( a_n \to 0 \) and \( b_n \) is bounded, does \( a_nb_n \to 0 \)? Proof or counterexample.

4. Let \( \{\vec{v}_j\} \) and \( \{\vec{w}_j\} \) be sequences of points in \( \mathbb{R}^3 \) and let \( \vec{v}_j \cdot \vec{w}_j \) be their inner (“dot”) product. If \( \vec{v}_j \to \vec{v} \) and \( \vec{w}_j \to \vec{w} \), show that \( \vec{v}_j \cdot \vec{w}_j \to \vec{v} \cdot \vec{w} \).

5. Let \( \vec{v} \) and \( \vec{w} \) be points in the plane, \( \mathbb{R}^2 \). Show that

\[
\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2 = 4\vec{v} \cdot \vec{w}.
\]

6. Let \( f : [0, 2] \to [0, 2] \) be a continuous function.

   a) Show that there is at least one point \( c \in [0, 2] \) so that \( f(c) = c \).

   b) Give an example showing that there may be more than one such point.

   c) If in addition \( f(x) \) is differentiable and \( |f'(x)| < 1 \) for all \( x \in [0, 2] \), show there is exactly one such point.

7. Examples:

   a) Give an example of a bounded set of real numbers that has exactly two limit points.

   b) Give an example of a bounded subset of \( \mathbb{R}^2 \) that contains all but one of its limit points.

   c) Give an example of an open cover of \( \{x \in \mathbb{R} : 0 < x \leq 1\} \) that has no finite sub-cover.

8. Let \( f : \{-1 < x < 2\} \to \mathbb{R} \) be a continuous function. If \( f(0) = 6 \), show there is some interval \( J := [-c, c] \subset \{-1 < x < 2\} \) with \( c > 0 \) so that \( f(x) > 3 \) for all \( x \in J \).
9. [CLASSIFY SETS] For each of the sets below, determine which of the following properties it has: open closed bounded compact countable

a) \{x \in \mathbb{R} : x = 1, 2, 3, 4\}

b) \{1 - \frac{1}{n} \in \mathbb{R}, \ n = 1, 2, 3, \ldots\}

c) \{1\} \cup \{1 + \frac{(-1)^n}{n} \in \mathbb{R}, \ n = 1, 2, 3, \ldots\}

d) \{x_n = (-1)^n + \frac{1}{n} \in \mathbb{R}, \ n = 1, 2, 3, \ldots\}

e) \{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are positive integers}\}

f) \{x \in \mathbb{R} : 0 < x < 1\}

g) \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, \ y = 0\}

h) \{x \in \mathbb{R} : 0 \leq x \leq 1\}

i) \{x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots\}

j) \{x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots\} \cup \{0\}

k) \{(x, y) \in \mathbb{R}^2 : 0 \leq y - x \leq 1\}

l) \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2\}

m) \{(x, y) \in \mathbb{R}^2 : x > 1, \ y < 1/x\}

n) \{x \in \mathbb{R} : x \text{ is a rational number}\}

o) \{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are rational numbers}\}

p) \{(k, n) \in \mathbb{R}^2 : k, n \text{ any positive integers with } k^2 + n^2 < 100\}

q) \{(x, y) \in \mathbb{R}^2 : x + y > 1\}

r) \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}

s) The following points in \mathbb{R}^4: \ e_1 = (1, 0, 0, 0), \ e_2 = (0, 1, 0, 0), \ e_3 = (0, 0, 1, 0), \ e_4 = (0, 0, 0, 1)\}

10. Let \(J\) be the interval \(J := \{0 \leq x \leq 1\}\).

a) Let \(f(x) = 0\) for all \(x \in J\) except that \(f(1/2) = 1\). Show that \(f\) is Riemann integrable on \(J\) and compute the value of the integral.

b) Say \(f(x)\) is continuous on \(J\) and \(f(x) \geq 0\) there. If \(\int_0^1 f(x) \, dx = 0\), show that \(f(x) = 0\) in \(J\).

c) Say \(f(x)\) is continuous in \(J\) and \(\int_0^1 f(x)g(x) \, dx = 0\) for every function \(g(x)\) that is continuous in \(J\). Show that \(f(x) = 0\) at every point of \(J\).

11. a) Let \(a_j \in \mathbb{R}\) be a sequence of real numbers with the property that

\[|a_{j+1} - a_j| \leq \frac{1}{2}|a_j - a_{j-1}|.\]

Show that \(|a_{j+1} - a_j| \leq 2^{-j}|a_1 - a_0|\).
Use this to show that $a_j$ is a Cauchy sequence and hence converge to some real number $A$.

**HINT:** If $n > k$, then

$$a_n - a_k = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{k+1} - a_k).$$

b) Let $\vec{v}_j \in \mathbb{R}^n$ be a sequence of points with the property that

$$\|\vec{v}_{j+1} - \vec{v}_j\| \leq \frac{1}{2}\|\vec{v}_j - \vec{v}_{j-1}\|.$$

Show that $\vec{v}_j$ is a Cauchy sequence and hence converge to some point $p \in \mathbb{R}^n$.

[Last revised: June 21, 2015]