DIRECTIONS This exam has three parts, Part A, short answer, has 1 problem (12 points). Part B has 5 shorter problems (7 points each, so 35 points). Part C has 3 traditional problems (15 points each so 45 points). Total is 92 points.
Closed book, no calculators or computers– but you may use one 3” × 5” card with notes on both sides.

Part A: Short Answer (1 problems, 12 points).

1. Let $S$ and $T$ be linear spaces and $A : S \rightarrow T$ be a linear map. Say $V$ and $W$ are particular solutions of the equations $AV = Y_1$ and $AW = Y_2$, respectively, while $Z \neq 0$ is a solution of the homogeneous equation $AZ = 0$.

Answer the following in terms of $V$, $W$, and $Z$.

a) Find some solution of $AX = 3Y_1$.

b) Find some solution of $AX = -5Y_2$.

c) Find some solution of $AX = 3Y_1 - 5Y_2$.

d) Find another solution (other than $Z$ and 0) of the homogeneous equation $AX = 0$.

e) Find two solutions of $AX = Y_1$.

f) Find another solution of $AX = 3Y_1 - 5Y_2$.

Part B: Short Problems (5 problems, 7 points each so 35 points)

B–1. $U = (1, 1, 0, 1)$ and $V = (-1, 2, 1, -1)$ are orthogonal vectors in $R^4$.

Write the vector $X = (1, 1, 1, 2)$ in the form $X = aU + bV + W$, where $a$, $b$ are scalars and $W$ is a vector perpendicular to $U$ and $V$.

B–2. Find $u(x,t)$ that satisfies $u_x - 2u_t = 1$ with $u(x,0) = 0$.

B–3. Let $u(x,t)$ be a solution of the wave equation

$$u_{tt} = 4u_{xx}, \quad \text{for} \quad -\infty < x < \infty, \ t \geq 0,$$

with the (continuous) initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Find the largest interval $J = \{a \leq x \leq b\}$ where changing $f(x)$ or $g(x)$ at any point of $J$ can change (“influence”) the value of $u(0,3)$. In other words, in the $(x,t)$ plane, find all the points on the $x$-axis that are in the domain of dependence of $(0, 3)$.
B–4. Find the general solution \( u(x, y) \) of \( u_{xy} = 4y \).

B–5. Let \( u(x, y) \) and \( v(x, y) \) be solutions of the Laplace equation \( \Delta u = 0, \Delta v = 0 \) in a bounded region \( \Omega \) in the plane. If \( u > v \) on the boundary of \( \Omega \), what, if anything, can you conclude about the relationship between \( u \) and \( v \) inside \( \Omega \)? Justify your assertion.

**Part C: Traditional Problems** (3 problems, 15 points each so 45 points)

C–1. Find the motion \( u(x, t) \) of a clamped string \( \{0 \leq x \leq \pi\} \)

\[
    u_{tt} = u_{xx},
\]

with initial and boundary conditions:

\[
    u(x, 0) = 0, \quad u_t(x, 0) = 15 \sin 5x, \quad \text{and} \quad u(0, t) = u(\pi, t) = 0.
\]

C–2. Let \( u(x, y) \) satisfy \( \Delta u - u = 0 \) in a bounded region \( \Omega \subset \mathbb{R}^2 \) with \( u(x, y) = 0 \) on the boundary of \( \Omega \). Use Green’s identity to show that \( u(x, y) = 0 \) throughout \( \Omega \).

C–3. Let \( u(x, t) \) be the temperature of a rod of length \( L \) that satisfies

\[
    u_t = u_{xx} - ru \quad \text{for} \quad 0 < x < L, \quad t > 0,
\]

where \( r > 0 \) is a constant [this is related to the heat equation but assumes that heat radiates out into the air along the rod]. Assume \( u \) satisfies the initial condition \( u(x, 0) = f(x) \).

Define the total heat energy by \( E(t) = \frac{1}{2} \int_0^L u^2(x, t) \, dx \).

a) If \( u \) also satisfies the Dirichlet boundary conditions

\[
    u(0, t) = 0, \quad u(L, t) = 0
\]

(the ends of the rod are held at temperature 0), show that \( E(t) \) is a decreasing function of \( t \).

b) Show that even if \( u \) satisfies Neumann boundary conditions

\[
    u_x(0, t) = 0, \quad u_x(L, t) = 0
\]

(the ends of the rod are insulated), \( E(t) \) is still a decreasing function of \( t \).

c) [Extra credit!] Show that in either of the above cases \( \lim_{t \to \infty} E(t) = 0 \).