PDE: Linear Change of Variable

Let $x := (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$ and consider the second order linear partial differential operator

$$Lu := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j},$$

where the coefficient matrix $A := (a_{ij})$ is constant. Since for functions whose second derivatives are continuous we know that

$$\frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_i} \right),$$

we may (and will) assume that $A$ is a symmetric matrix: $A = A^*$. In these brief notes we obtain a useful formula for how $L$ changes if we make the linear change of variable $y = Sx$ where $(S := s_{ij})$ is a constant matrix. Written in coordinates this means that

$$y_k = \sum_{\ell=1}^{n} s_{k\ell} x_{\ell}, \quad \text{where} \quad k = 1, \ldots, n.$$

**FIRST GOAL:** Compute $L$ in these new $y$ coordinates. This is straightforward (even boring) if you just keep calm and don’t make copying errors. By the chain rule

$$\frac{\partial u}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} s_{k\ell}.$$  

(2)

We repeat this process to compute the second derivatives:

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_j} \right) = \sum_{k=1}^{n} \frac{\partial}{\partial y_k} \left( \sum_{l=1}^{n} \frac{\partial u}{\partial y_l} s_{kl} \right) = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial y_k \partial y_l} s_{k\ell} s_{l\ell}.$$  

(3)

Consequently

$$Lu = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l=1}^{n} \left[ \sum_{i,j=1}^{n} a_{ij} s_{k\ell} s_{l\ell} \right] \frac{\partial^2 u}{\partial y_k \partial y_l}.$$  

so

$$Lu = \sum_{k,l=1}^{n} b_{k\ell} \frac{\partial^2 u}{\partial y_k \partial y_l},$$

where the coefficient matrix $B := (b_{k\ell})$ is

$$b_{k\ell} = \sum_{i,j=1}^{n} a_{ij} s_{k\ell} s_{l\ell}.$$  

In terms of matrices this simply says that

$$B = SAS^*.$$  

(4)

**SECOND GOAL:** Pick the matrix $S$ defining the change of coordinates $y = Sx$ to make (3) as simple as possible. We’ll be able to make $B$ into a diagonal matrix by diagonalizing $A$. Since $A$ is a symmetric matrix, there is an orthogonal matrix $R$ that diagonalizes it (in $\mathbb{R}^n$, an orthogonal matrix is just the generalization of a rotation). Thus

$$R^{-1}AR = \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$  

Thus $S = RAR^{-1}$. Since for an orthogonal matrix $R$ we know that $R^* = R^{-1}$, we let $S = R^*$, then $\Lambda = SAS^*$. Comparing with (4) we see that using this change of coordinates we have arranged that $B$ is a diagonal matrix. Consequently $L$ has the much simpler form

$$Lu = \lambda_1 \frac{\partial^2 u}{\partial y_1^2} + \lambda_2 \frac{\partial^2 u}{\partial y_2^2} + \cdots + \lambda_n \frac{\partial^2 u}{\partial y_n^2}.$$  

(5)

We can make one further simplification. By stretching the coordinates to have the coefficients in (5) be either 1, 0, or -1. For instance, if $\lambda_1 > 0$, replace $y_1$ by the new stretched coordinate $z_1 := y_1 / \sqrt{\lambda_1}$. As an example, using this device

$$Lu := 4 \frac{\partial^2 u}{\partial y_1^2} - 9 \frac{\partial^2 u}{\partial y_2^2}$$  

becomes

$$Lu := \frac{\partial^2 u}{\partial z_1^2} - \frac{\partial^2 u}{\partial z_2^2}.$$  

**EXERCISE:** Show that there is a linear change of variable so that at one point, say the origin, the second derivative matrix

$$\frac{\partial^2 u}{\partial y_2 \partial y_1}(0)$$