1. Let $u(t)$ be the amount of a radioactive element at time $t$ and say initially, $u(0) = A$. The rate of decay is proportional to the amount present, so

$$\frac{du}{dt} = cu(t),$$

where the constant $c$ determines the decay rate. The half-life $T$ is the amount of time for half of the element to decay, so $u(T) = \frac{1}{2} u(0)$. Find $c$ in terms of $T$ and obtain a formula for $u(t)$ in terms of $T$.

2. Let $\int_0^x f(t) \, dt = e^{\cos(3x+1)} + A$, where $f$ is some continuous function. Find $f$ and the constant $A$.

3. Say $w(t)$ satisfies the differential equation

$$aw''(t) + bw' + cw(t) = 0,$$

where $a$ and $c$, are positive constants and $b \geq 0$. Let $E(t) = \frac{1}{2}[aw^2 + cw^2]$.

a) Without solving the differential equation, show that $E'(t) \leq 0$.

b) Use this to show that If you also know that $w(0) = 0$ and $w'(0) = 0$, then $w(t) = 0$ for all $t \geq 0$.

c) [Uniqueness] Say the functions $u(t)$ and $v(t)$ both satisfy the same equation (1) and also $u(0) = v(0)$ and $u'(0) = v'(0)$. Show that $u(t) = v(t)$ for all $t \geq 0$.

4. Say $u(x,y)$ has the property that $\frac{\partial u}{\partial y} = 0$ for all points $(x,y)$ and that $u(x,0) = \sin 3x$. Find $u(x,y)$.

What if instead $u$ satisfies $\frac{\partial u}{\partial y} = 2xy$ ?

5. A function $u(x,y)$ satisfies $u_x + 3u_y = 0$. Find a change of variables

$$x = as + bt$$

$$y = cs + dt$$

so that in the new $(s,t)$ variables $u$ satisfies $\frac{\partial u}{\partial s} = 0$.

[Last revised: January 15, 2011]