Problem Set 1

Due: In class Thursday, Jan. 27. Late papers will be accepted until 1:00 PM Friday.

1. Find Green’s function \( g(x,s) \) to get a formula \( u(x) = \int_0^x g(x,s)f(s)\,ds \) for a particular solution of \( u''(x) = f(x) \).

2. In class we considered the oscillations of a weight attached to a spring hanging from the ceiling. If \( u(t) \) is the displacement of the mass \( m \) we were let to solve \( mu''(t) = -ku \), where \( k > 0 \) is a constant that depends on the stiffness of the spring. But this model neglected gravity. If we include gravity the equation becomes
   \[ mu'' = -ku + mg, \]
   where \( g \) is the gravitational constant,
   Solve this equation assuming you know the initial conditions \( u(0) = A \) and \( u'(0) = B \).

3. Let \( a(x) \) and \( f(x) \) be periodic functions with period \( P \), so, for instance, \( a(x + P) = a(x) \). This problem investigates periodic solutions \( u(x) \) (with period \( P \)) of \( Lu := u'(x) + a(x)u = f(x) \).
   a) Show there is a periodic solution of \( u'(x) = f(x) \) if and only if \( \int_0^P f(x)\,dx = 0 \).
   b) Show that the homogeneous equation \( Lu = 0 \) has a non-trivial \( P \)-periodic solution \( u(x) \) if and only if \( \int_0^P a(x)\,dx = 0 \).
   c) If \( \int_0^P a(x)\,dx \neq 0 \), show that the inhomogeneous equation \( Lu = f \) always has a unique \( P \)-periodic solution \( u(x) \).
      On the other hand, if \( \int_0^P a(x)\,dx = 0 \), find a necessary and sufficient condition for \( Lu = f \) to have a \( P \)-periodic solution. If it has a \( P \) periodic solution, is this solution unique?

4. In class we obtained a simpler general formula for a particular solution of the inhomogeneous first order system \( U' + AU = F \), where \( U(x) \) and \( F(x) \) are vectors with \( n \) components and \( A(x) \) is an \( n \times n \) matrix. In addition we showed how much of the theory for a second order equation was in fact a special case of that for a first order system.
   Use this to re-derive Lagrange’s formula for a particular solution of the inhomogeneous equation \( u'' + u = f \).

5. Show that the boundary value problem \( u'' + u = f \) on \( 0 \leq x \leq \pi \) with boundary conditions \( u(0) = 0 \) and \( u(\pi) = 0 \) has a solution if and only if \( \int_0^\pi f(x)\sin x\,dx = 0 \).

Bonus Problems (Due Jan. 27)

1-B Let \( a(t) \) and \( f(t) \) be periodic continuous functions with period \( 2\pi \).
a) Show that the equation \( u'' = f \) has a \( 2\pi \)-periodic solution (so both \( u \) and \( u' \) are \( 2\pi \)-periodic) if and only if
\[
\int_0^{2\pi} f(t) \, dt = 0.
\]

b) Show that the equation \( u'' + u = f \) has a \( 2\pi \)-periodic solution if and only if both
\[
\int_0^{2\pi} f(t) \sin t \, dt = 0 \quad \text{and} \quad \int_0^{2\pi} f(t) \cos t \, dt = 0.
\]

c) More generally, show that the equation \( Lu := u'' + a(t)u = f \) has a \( 2\pi \)-periodic solution if and only if
\[
\int_0^{2\pi} f(t)z(t) \, dt = 0
\]
for all \( 2\pi \)-periodic solutions of \( z'' + a(t)z = 0 \). [REMARK: These are special cases of the Fredholm alternative: the image of \( L \) is the orthogonal complement of the kernel of the adjoint operator \( L^* \).]

[Last revised: January 25, 2011]