1. In \( \mathbb{R}^4 \) the vectors

\[
U_1 := (1, 1, 1, 1), \quad U_2 := (1, 1, -1, -1), \quad U_3 := (2, -2, 2, 2), \quad U_4 := (1, -1, -1, 1)
\]

are orthogonal, as you can easily verify.

a) Use these to find an orthonormal basis \( e_k := \alpha_k U_k, \quad k = 1, \ldots, 4 \).

b) Write the vector \( v := (0, -2, 2, 5) \) using this basis: \( v = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 \).

c) Find the projection, \( P_v \), of \( v \) into the plane spanned by \( U_2 \) and \( U_3 \).

d) Compute \( \|P_v\| \).

2. Let \( X \) be a linear space with an inner product (not necessarily \( \mathbb{R}^n \)) and let \( P : X \rightarrow X \) be an orthogonal projection, so \( P^2 = P \) and \( P = P^\ast \). Write \( V \) for the image of \( P \); it is the space into which vectors are projected. Given \( x \in X \), write \( x = v + w \), where \( v = Px \) is the projection of \( x \) into \( V \). Show that \( w \) is orthogonal to \( V \).

3. Let \( f(x) \) be a \( 2\pi \) periodic function. Use Fourier series to investigate finding \( 2\pi \) periodic solutions of

\[
-u''(x) + u = f(x),
\]

so we want \( u \) and all of its derivatives to be \( 2\pi \) periodic.

This is routine – and short. Expand \( f \) in a Fourier series, so \( f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} \) and seek the solution as a Fourier series \( u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \). So all you need do is determine the \( c_k \)'s in terms of the \( a_k \)'s.

4. Consider the wave equation \( u_{tt} = u_{xx} \), \( 0 \leq x \leq \pi \) with the boundary conditions

\[
u(0, t) = 0, \quad u(\pi, t) = 0, \quad (t \geq 0).
\]

a) Find all solutions of the special form \( u(x, t) = \phi(x)T(t) \) (standing wave solutions).

b) Use this to solve the wave equation with the above boundary conditions and the initial conditions

\[
u(x, 0) = 2 \sin(3x) - 7 \sin(19x), \quad u_t(x, 0) = 0.
\]
5. Consider the wave equation \( u_{tt} = u_{xx}, \) \( 0 \leq x \leq \pi \) with the mixed boundary conditions

\[
  u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0, \quad (t \geq 0).
\]

a) Find all solutions of the special form 
\( u(x,t) = \phi(x)T(t) \) (standing wave solutions).

b) Use this to solve the wave equation with the above boundary conditions and the initial conditions
\( u(x,0) = 4 \sin(\frac{5x}{2}) - 7 \sin(\frac{9x}{2}), \quad u_t(x,0) = 0. \)

6. Lorentz Transformations  Let \( u(x,t) \) be a given function. Find all linear changes of variable
\[
  \tau = \alpha x + \beta t, \quad z = \gamma x + \delta t
\]
that keep the wave operator invariant, that is
\[
  u_{tt} - c^2 u_{xx} = u_{\tau\tau} - c^2 u_{zz}.
\]

**Suggestion:** You will be led to three equations for the four coefficients. Try to find a cleaner way to write these in terms of some other parameter. Here is a related example. Say \( a, b, c, \) and \( d \) satisfy
\[
  a^2 + b^2 = 1, \quad c^2 + d^2 = 1, \quad ac + bd = 0. \tag{1}
\]
In this example, try writing \( a = \cos \theta \). Then \( b = \pm \sin \theta \) etc and you’ll get equations for the four coefficients in terms of the one parameter \( \theta \) (with some choices for \( \pm \) a few places?). Upshot, the equations (1) just describe a rotation (and possibly also a reflection) around the origin in the plane \( \mathbb{R}^2 \).

7. Integration by Parts for Multiple Integrals  Let \( u(x,y) \) be a scalar function and \( \mathbf{F}(x,y) \) a vector field in a bounded region \( D \) in \( \mathbb{R}^2 \) and let the closed curve \( C \) be the boundary of \( D \) with \( \mathbf{N} \) be the unit outer normal vector field on this boundary.

a) Prove the identity \( \nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u\nabla \cdot \mathbf{F} \). Compare this with the special case of a function of one variable.

b) Use the divergence theorem to obtain the following generalization of integration by parts for multiple integrals:
\[
\iint_D u\nabla \cdot \mathbf{F} \, dA = \oint_C u\mathbf{F} \cdot \mathbf{N} \, ds - \iint_D \nabla u \cdot \mathbf{F} \, dA.
\]
Notice that for a function of one variable with \( D \) being the interval \( \{a < x < b\} \), this reduces precisely to the usual formula for integration by parts.

c) Generalize this formula to the case where \( D \) is a bounded (solid) region in three dimensional space.
d) One frequently uses this with \( F = \nabla v \). Show the above formula for integration by parts becomes (say in two dimensions)

\[
\iint_D u \Delta v \, dA = \oint_C u \nabla v \cdot \mathbf{N} \, ds - \iint_D \nabla u \cdot \nabla v \, dA.
\]

To what does this reduce for functions on one variable?

e) As a short application using this, say \( u(x,y) \) is a harmonic function in a bounded region \( D \), so \( \nabla \cdot \nabla u = 0 \). One can think of \( u(x,y) \) as being the equilibrium temperature of \( D \). Let \( C \) is the boundary of \( D \). If \( u = 0 \) on \( C \), it is plausible that one must have \( u(x,y) = 0 \) throughout \( D \). Show how this follows from the above formula. What is the analogous assertion for functions of one variable, where a harmonic function is just a solution of \( u'' = 0 \)?
Bonus Problem

1-B [Fourier Series in Several Variables]. Fourier series extends immediately to functions of several variables. Let $T^2$ be the square $\{(x,y) \in \mathbb{R}^2 \mid -\pi \leq x \leq \pi, -\pi \leq y \leq \pi\}$ and consider functions $f(x,y)$ that are $2\pi$ periodic in both variables with the $L_2(T^2)$ inner product

$$\langle f, g \rangle := \iint_{T^2} f(x,y)\overline{g(x,y)} \, dx \, dy.$$

a) Show that the functions

$$\varphi_{jk} := e^{i(jx+ky)} \quad j, k = 0, \pm 1, \pm 2, \ldots$$

are orthogonal. How should you modify these to get orthonormal functions?

b) If $f(x,y)$ is $2\pi$ periodic in both variables, use Fourier series to investigate finding periodic solutions $u(x,y)$ of

$$-\Delta u(x,y) + u = f(x,y).$$

[This is almost identical to Problem 3 above.]

c) If $f(x,y)$ is $2\pi$ periodic in both variables, use Fourier series to investigate finding periodic solutions of

$$-\Delta u(x,y) = f(x,y).$$

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